THE QUARTIC FORMULA

We want the solutions to

$$x^4 + px^3 + qx^2 + rx + s = 0\ , \tag{1}$$

where p, q, r and s are real. In what follows, let

$$d = \tfrac{1}{8}(8q - 3p^2)\ , \quad e = \tfrac{1}{8}(8r - 4pq + p^3)\ , \quad f = \tfrac{1}{256}(256s - 64pr + 16p^2q - 3p^4)\ . \tag{2}$$

The solutions to eqn. (1) follow two cases.

Case 1; $e = 0$:

Let y be the four roots of

$$y^4 + dy^2 + f = 0\ , \tag{3}$$

which is quadratic in y^2. In other words, use the Quadratic Formula to calculate two values of y^2, and then take the two square roots of each of the y^2-values. The four solutions to eqn. (1) are then

$$x = -\tfrac{1}{4}p + y\ . \tag{4}$$

Case 2; $e \neq 0$:

For this case, additionally, let

$$D = \tfrac{1}{3456}(2d^3 - 72df + 27e^2)\ , \tag{5a}$$

$$E = \tfrac{1}{442{,}368}(128d^2f^2 + 27e^4 - 16d^4f + 4d^3e^2 - 144de^2f - 256f^3)\ , \tag{5b}$$

$$F = \tfrac{1}{144}(d^2 + 12f)\ . \tag{5c}$$

Now, in what follows, all roots are either the principal root or the first complex root. There is the possibility that G (and thus A , B and C) directly below may be complex. Notwithstanding, calculate

$$G = \sqrt[3]{D + \sqrt{E}} \tag{6}$$

and then

$$A = \sqrt{-\tfrac{1}{6}d + G + \tfrac{F}{G}} \qquad \text{if} \qquad F \neq 0 \tag{7a}$$

or

$$A = \sqrt{-\tfrac{1}{6}d + G} \qquad \text{if} \qquad F = 0\ . \tag{7b}$$

Next, calculate

$$B = \sqrt{A^2 + \tfrac{1}{2}d + \tfrac{e}{4A}}\ , \qquad C = \sqrt{A^2 + \tfrac{1}{2}d - \tfrac{e}{4A}}\ . \tag{8}$$

Finally then, the four solutions to eqn. (1) are

$$x = -\tfrac{1}{4}p + A \pm iB\ , \tag{9a}$$

$$x = -\tfrac{1}{4}p - A \pm iC\ , \tag{9b}$$

where $i = \sqrt{-1}$.