ROOTS OF POLYNOMIALS

In what follows, the coefficients p, q, r and s of polynomial $P(x) = 0$ are real-valued.

QUADRATICS

The roots of $P(x) = x^2 + px + q = 0$ are given by the Quadratic Formula, *viz.*,

$$x = -\tfrac{1}{2}p \pm \tfrac{1}{2}\sqrt{p^2 - 4q}\ . \tag{1}$$

In eqn. (1), the square root is the principal square root.

CUBICS

The roots of $P(x) = x^3 + px^2 + qx + r = 0$ follow three cases. Let

$$c = \tfrac{1}{9}(3q - p^2)\ , \qquad d = \tfrac{1}{54}(27r - 9pq + 2p^3)\ . \tag{2}$$

The discriminant is then

$$D = d^2 + c^3. \tag{3}$$

Case 1: $D = 0$. There are three real roots, at least one of which occurs twice, *viz.*,

$$\begin{aligned} x &= -\tfrac{1}{3}p + \sqrt[3]{d} \quad \text{(twice)}\ , \\ x &= -\tfrac{1}{3}p - 2\sqrt[3]{d} \quad \text{(once)}\ . \end{aligned} \tag{4}$$

In eqns. (4), the cube roots are real-valued.

Case 2: $D > 0$. There is one real root and a pair of complex conjugates. Let

$$A^3 = -d + \sqrt{D}, \qquad B^3 = -d - \sqrt{D}, \tag{5}$$

where the square root is the principal square root. Next, find A and B by taking real cube roots. Finally then, the three roots of $P(x) = 0$ are

$$\begin{aligned} x &= -\tfrac{1}{3}p + A + B \quad \text{(real)}\ , \\ x &= -\tfrac{1}{3}p - \tfrac{1}{2}(A + B) \pm \tfrac{\sqrt{3}}{2}(A - B)i \quad \text{(complex conjugates)}\ . \end{aligned} \tag{6}$$

Case 3: $D < 0$. There will be three real roots. Let

$$A^3 = -d + i\sqrt{-D}, \tag{7}$$

in which the square root is the principal square root and A^3 is complex. Next, calculate A as the first complex cube root by writing A^3 in polar form

$$A^3 = r(\cos\theta + i\sin\theta), \qquad r > 0\,, \qquad 0° < \theta < 180° \tag{8}$$

so that

$$A = \sqrt[3]{r}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right). \tag{9}$$

An alternate procedure for calculating the first complex cube root is just to use your graphing calculator, *e.g.*, $(-2+2i)$^$(1/3) = 1 + i$. Finally, and in any case, the three roots of $P(x) = 0$ are

$$x = -\frac{1}{3}p + 2(\mathrm{Re}A) \quad \text{(one real root)},$$
$$x = -\frac{1}{3}p - (\mathrm{Re}A) \pm \sqrt{3}(\mathrm{Im}A) \quad \text{(two real roots)}, \tag{10}$$

in which $(\mathrm{Re}A)$ and $(\mathrm{Im}A)$ are the real and imaginary parts of A.

QUARTICS

The roots of $P(x) = x^4 + px^3 + qx^2 + rx + s = 0$ follow two cases. Let

$$d = \frac{1}{8}(8q - 3p^2)\,, \qquad e = \frac{1}{8}(8r - 4pq + p^3)\,, \qquad f = \frac{1}{256}(256s - 64pr + 16p^2q - 3p^4)\,. \tag{11}$$

Case 1: $e = 0$. Let y be the four roots of

$$y^4 + dy^2 + f = 0, \tag{12}$$

which may be solved for two values of y^2 via the Quadratic Formula. The four values of y are then obtained by taking the two (possibly complex) square roots of each of the y^2 values. The four solutions to $P(x) = 0$ are then

$$x = -\frac{1}{4}p + y. \tag{13}$$

Case 2: $e \neq 0$. Let A^2 be a real root of

$$A^6 + \frac{d}{2}A^4 + \frac{1}{4}\left(\frac{d^2}{4} - f\right)A^2 - \frac{e^2}{64} = 0, \tag{14}$$

which equation is cubic in A^2, and let A be the principal square root of A^2. Thus, A may be real ($A^2 > 0$) or purely imaginary ($A^2 < 0$). Also let

$$B^2 = A^2 + \frac{1}{2}d + \frac{e}{4A}\,, \qquad C^2 = A^2 + \frac{1}{2}d - \frac{e}{4A}\,, \tag{15}$$

where B^2 and C^2 may be either real ($A^2 > 0$) or complex ($A^2 < 0$). Note that if B^2 and C^2 are complex, then they are conjugates. Next, obtain B and C as either a principal square root (if B^2 and C^2 are real) or as the first complex square root (if B^2 and C^2 are complex). In other words, B and C may be real, purely imaginary or complex. If B and C are complex, then they are conjugates. In any case, the four solutions to $P(x) = 0$ are then

$$x = -\frac{1}{4}p + A \pm iB,$$
$$x = -\frac{1}{4}p - A \pm iC. \tag{16}$$

QUINTICS AND HIGHER

It can be shown that no formulas, such as those above, can be derived for the solutions of quintic and higher order polynomials. Iterative methods must be used.