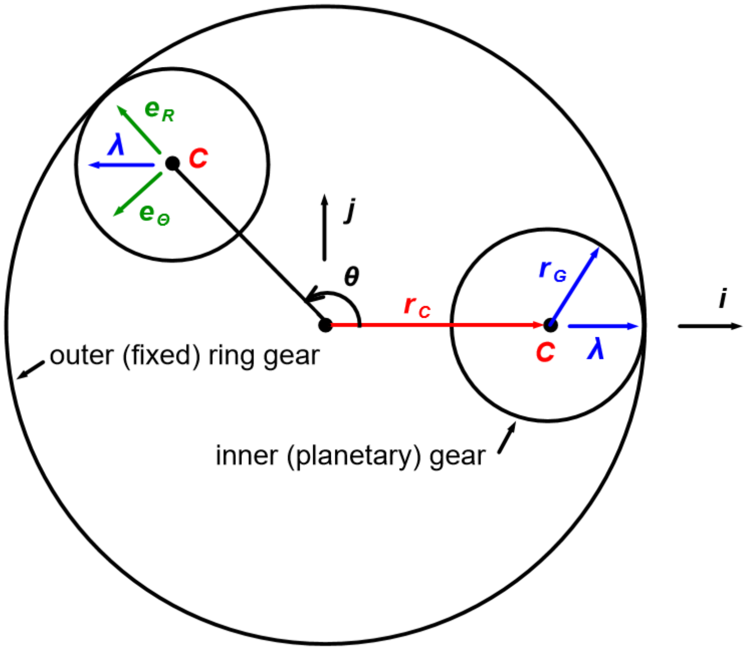
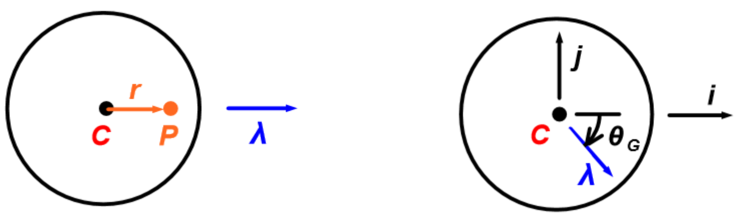
PARAMETRIC EQUATIONS FOR SPIROGRAPH CURVES

Below is shown two gears, *i.e.*, an outer (fixed) ring gear and an inner (planetary) gear, which are used to draw spirograph curves.



In the figure, r_G is the radius of the planetary gear, $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along the *x*– and *y*–directions, $\boldsymbol{\lambda}$ is a unit vector which is embedded in (*i.e.*, which rotates with) the planetary gear, $\boldsymbol{e}_\mathrm{R}$ and $\boldsymbol{e}_\Theta$ are unit vectors in the radial and tangential directions, and θ is a parameter which will sweep out the curve. Note that the radius of the ring gear is $r_C + r_G$.

The figures directly below are of the planetary gear. In the left-hand figure (which corresponds to $\theta_G = 0$), the point *P* represents the position of the pen. In the right-hand figure, θ_G measures the orientation of the planetary gear.



The curve is then traced out by moving θ in the counterclockwise direction, with the planetary gear rolling (without sliding) along the ring gear.

Due to the gear action, $(r_C + r_G)\theta = r_G\theta_G$, or

$$\theta_G = \left(1 + \frac{r_C}{r_G}\right)\theta\,. \tag{1}$$

Also, consistent with the above figures (and with the gear action), one sees,

$$\boldsymbol{\lambda} = \cos\theta_G\,\boldsymbol{e}_R - \sin\theta_G\,\boldsymbol{e}_\Theta \tag{2}$$

and

$$\begin{aligned}\boldsymbol{e}_R &= \quad \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}\\ \boldsymbol{e}_\Theta &= -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}\,.\end{aligned} \tag{3}$$

Substitution of eqns. (3) into eqn. (2) gives, after using the angle difference identities for sine and cosine, that $\boldsymbol{\lambda} = \cos(\theta_G - \theta)\,\mathbf{i} - \sin(\theta_G - \theta)\,\mathbf{j}$, or via eqn. (1),

$$\boldsymbol{\lambda} = \cos\left(\frac{r_C}{r_G}\theta\right)\mathbf{i} - \sin\left(\frac{r_C}{r_G}\theta\right)\mathbf{j}\,. \tag{4}$$

Now, the absolute position $\mathbf{r}_P$ of the pen P is $\mathbf{r}_P = \mathbf{r}_C + \mathbf{r}_{P/C}$, where $\mathbf{r}_C = r_C\boldsymbol{e}_R$ is the position of the center point C of the planetary gear, and $\mathbf{r}_{\mathrm{P/C}} = r\boldsymbol{\lambda}$ is the position of the pen relative to point C. So, finally, writing $\mathbf{r}_P = x\,\mathbf{i} + y\,\mathbf{j}$, and then by using eqns. (3) and (4), one obtains the parametric equations of the spirograph curve

$$x = r_C\cos\theta + r\cos\left(\frac{r_C}{r_G}\theta\right) \tag{5}$$

$$y = r_C\sin\theta - r\sin\left(\frac{r_C}{r_G}\theta\right).$$

The range of values that θ must traverse in order for the spirograph curve to close upon itself is found by writing the ratio r_C/r_G as a reduced (proper or improper) fraction. The number of revolutions that θ must complete is then the denominator of this fraction.