THE CUBIC FORMULA

We want to solve

$$P(x) = x^3 + px^2 + qx + r = 0 \tag{1}$$

for x, where p, q and r are real. First, deflate eqn. (1) by substituting

$$x = y - \tfrac{1}{3}p \tag{2}$$

into eqn. (1). The result is

$$P(y) = y^3 + 3cy + 2d = 0 \tag{3}$$

where

$$c = \tfrac{1}{9}(3q - p^2), \qquad d = \tfrac{1}{54}(27r - 9pq + 2p^3). \tag{4}$$

Equation (3) can be solved by assuming a solution of the form

$$y = \left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right)A + \left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)B. \tag{5}$$

Note that $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ are the two complex-valued cube roots of one. In any case, cube each side of eqn. (5) to obtain

$$y^3 = A^3 + B^3 + 3AB\left\{\left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right)A + \left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)B\right\},$$

or via eqn. (5),

$$y^3 = A^3 + B^3 + 3ABy,$$

or rearranging,

$$y^3 - 3ABy - (A^3 + B^3) = 0. \tag{6}$$

Comparing eqns. (3) and (6), one sees

$$3c = -3AB, \quad 2d = -(A^3 + B^3), \tag{7a,b}$$

or inversely,

$$AB = -c, \qquad A^3 + B^3 = -2d. \tag{8a,b}$$

Now, solve eqns. (8). So, substitute $B = -\frac{c}{A}$ into eqn. (8b) to obtain

$$A^6 + 2dA^3 - c^3 = 0, \tag{9a}$$

which is quadratic in A^3. Similarly, substitute $A = -\frac{c}{B}$ into eqn. (8b) to obtain

$$B^6 + 2dB^3 - c^3 = 0, \tag{9b}$$

which is quadratic in B^3. Looking at eqns. (8), one sees that if A and B are complex, then they are conjugates (in which case $c < 0$, *cf.*, eqn. 8a, and y is real, *cf.*, eqn. 5). In any case, motivated by the fact that A and B may be conjugates, via the Quadratic Formula, one of the solutions to eqns. (9) is

$$A^3 = -d + \sqrt{D}, \qquad B^3 = -d - \sqrt{D} \tag{10}$$

where

$$D = d^2 + c^3. \tag{11}$$

Case 1: $D = 0$.

Here, $A^3 = B^3 = -d$, or $A = B = -\sqrt[3]{d}$, which when put into eqn. (5) gives

$$y = \sqrt[3]{d} \tag{12}$$

as one of the roots of eqn. (3). In this case, eqn. (3) can be factored as

$$(y^3 + 3cy + 2d) \div \left(y - \sqrt[3]{d}\right) = Q(y) = y^2 + \sqrt[3]{d}y - 2\left(\sqrt[3]{d}\right)^2, \tag{13}$$

which result is obtained by long division and eqn. (8a). The other two roots of eqn. (3) are then obtained by solving $Q(y) = 0$. The Quadratic Formula gives

$$y = \sqrt[3]{d}, \qquad y = -2\sqrt[3]{d}. \tag{14}$$

Note that the three roots (eqns. 12 and 14) of the deflated polynomial (3) add to zero. Notwithstanding, the three roots of eqn. (1), in this case, follow from eqn. (2), *viz.*,

$$x = -\tfrac{1}{3}p + \sqrt[3]{d} \quad \text{(twice)}$$

$$x = -\tfrac{1}{3}p - 2\sqrt[3]{d} \quad \text{(once)}. \tag{15}$$

So, in this case, there are three real roots.

Case 2: $D > 0$.

Here A and B are real, so calculate A and B as the real cube roots of A^3 and B^3, *cf.*, eqn. (10). Rearranging eqn. (5), one of the roots to eqn. (3) is

$$y = -\tfrac{1}{2}(A + B) + \tfrac{\sqrt{3}}{2}(A - B)i, \tag{16a}$$

and another root must be the conjugate, *i.e.*,

$$y = -\tfrac{1}{2}(A + B) - \tfrac{\sqrt{3}}{2}(A - B)i. \tag{16b}$$

Now, the roots (16) satisfy the quadratic

$$\left[y - \left\{-\tfrac{1}{2}(A + B) + \tfrac{\sqrt{3}}{2}(A - B)i\right\}\right]\left[y - \left\{-\tfrac{1}{2}(A + B) - \tfrac{\sqrt{3}}{2}(A - B)i\right\}\right] =$$

$$= y^2 + (A + B)y + (A^2 - AB + B^2) = 0. \tag{17}$$

Via eqn. (17), eqn. (3) can be factored as

$$(y^3 + 3cy + 2d) \div \{y^2 + (A + B)y + (A^2 - AB + B^2)\} = L(y) = y - (A + B), \tag{18}$$

which result is obtained via long division and eqns. (8). So, the third root is

$$y = A + B. \tag{19}$$

Again, note that the three roots (eqns. 16 and 19) of the deflated polynomial (3) add to zero. Finally, these three roots, when put into eqn. (2), yield the three roots to eqn. (1), *viz.*,

$$x = -\tfrac{1}{3}p + A + B \qquad \text{(real)},$$

$$x = -\tfrac{1}{3}p - \tfrac{1}{2}(A + B) \pm \tfrac{\sqrt{3}}{2}(A - B)i \qquad \text{(complex conjugates)}. \tag{20}$$

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Case 3: $D < 0$.

Here A and B are complex conjugates. So, calculate A as the first complex cube root of

$$A^3 = -d + i\sqrt{-D}, \tag{21}$$

cf., eqn. (10), and $B = \bar{A}$ (the over-bar means conjugate). In this case, eqn. (5) becomes

$$y = -\tfrac{1}{2}(A + \bar{A}) + \tfrac{\sqrt{3}}{2}(A - \bar{A})i,$$

or since $\frac{1}{2}(A + \bar{A}) = \mathrm{Re}A$ and $\frac{1}{2}(A - \bar{A})i = -\mathrm{Im}A$,

$$y = -\mathrm{Re}A - \sqrt{3}\mathrm{Im}A \tag{22}$$

is a root of eqn. (3), where ReA and ImA mean, respectively, the real and imaginary parts of A. Now, when $B = \bar{A}$, eqns. (8) can be rewritten as

$$(\mathrm{Re}A)^2 + (\mathrm{Im}A)^2 = -c, \qquad 2\mathrm{Re}A\{(\mathrm{Re}A)^2 - 3(\mathrm{Im}A)^2\} = -2d. \tag{23}$$

Also, owing to the root (22), eqn. (3) may be factored, via long division and eqns. (23), as

$$(y^3 + 3cy + 2d) \div \{y - (-\mathrm{Re}A - \sqrt{3}\mathrm{Im}A)\} = Q(y)$$

$$Q(y) = y^2 - (\mathrm{Re}A + \sqrt{3}\mathrm{Im}A)y - 2\mathrm{Re}A(\mathrm{Re}A - \sqrt{3}\mathrm{Im}A).$$

Solving $Q(y) = 0$ via the Quadratic Formula then supplies the other two roots as

$$y = 2\mathrm{Re}A, \qquad y = -\mathrm{Re}A + \sqrt{3}\mathrm{Im}A. \tag{24}$$

Again, note that the three roots (eqns. 22 and 24) of the deflated polynomial (3) add to zero. Finally, the three roots, when put into eqn. (2), yield the three roots of eqn. (1), *i.e.*,

$$y = -\tfrac{1}{3}p + 2\mathrm{Re}A, \qquad y = -\tfrac{1}{3}p - \mathrm{Re}A \pm \sqrt{3}\mathrm{Im}A. \tag{25}$$

So, in this case, there are three real roots.