THE QUARTIC FORMULA

We want to solve

$$P(x) = x^4 + px^3 + qx^2 + rx + s = 0 \tag{1}$$

for x, where p, q, r and s are real. First, deflate eqn. (1) by substituting

$$x = y - \tfrac{1}{4}p \tag{2}$$

into eqn. (1). The result is

$$P(y) = y^4 + dy^2 + ey + f = 0, \tag{3}$$

where

$$d = \tfrac{1}{8}(8q - 3p^2), \qquad e = \tfrac{1}{8}(8r - 4pq + p^3), \qquad f = \tfrac{1}{256}(256s - 64pr + 16p^2q - 3p^4). \tag{4}$$

Case 1: $e = 0$.

In this case, eqn. (3) is

$$P(y) = y^4 + dy^2 + f = 0, \tag{5}$$

which is quadratic in y^2. Consequently, use the Quadratic Formula to solve for two values of y^2. Next, take the two (possibly complex) square roots of each of the y^2 values to obtain four y values. Finally, knowing the four roots of eqn. (5), the four roots of eqn. (1) follow directly from eqn. (2).

Case 2: $e \neq 0$.

Equation (3) can be solved by assuming four roots of the form

$$y = A \pm iB, \qquad y = -A \pm iC. \tag{6}$$

These are motivated by the fact that the four roots of the deflated polynomial add to zero, and that one possible scenario is for there to be two pairs of complex conjugate roots.

Notwithstanding, assuming that A, B and C are real, substitution of eqns. (6) into eqn. (3) yields

$$\begin{aligned} P(A + iB) = \{&A^4 + B^4 - 6A^2B^2 + dA^2 - dB^2 + eA + f\} \\ &+ iB\{2A(2A^2 - 2B^2 + d) + e\} = 0 \end{aligned} \tag{7}$$

and

$$\begin{aligned} P(-A + iC) = \{&A^4 + C^4 - 6A^2C^2 + dA^2 - dC^2 - eA + f\} \\ &+ iC\{-2A(2A^2 - 2C^2 + d) + e\} = 0. \end{aligned} \tag{8}$$

The expressions for $P(A - iB) = 0$ and $P(-A - iC) = 0$ are simply the conjugates of eqns. (7) and (8), respectively. Again, assuming that A, B and C are real (they don't have to be in the end, though), set the real and imaginary parts of eqns. (7) and (8) to zero (which solves eqns. 7 and 8 and their conjugates). Thus,

$$A^4 + B^4 - 6A^2B^2 + dA^2 - dB^2 + eA + f = 0 \tag{9}$$

$$A^4 + C^4 - 6A^2C^2 + dA^2 - dC^2 - eA + f = 0 \tag{10}$$

$$2A(2A^2 - 2B^2 + d) + e = 0 \tag{11}$$

$$-2A(2A^2 - 2C^2 + d) + e = 0. \tag{12}$$

Equations (11) and (12), respectively, give

$$B^2 = A^2 + \frac{d}{2} + \frac{e}{4A} \tag{13}$$

and

$$C^2 = A^2 + \frac{d}{2} - \frac{e}{4A}. \tag{14}$$

Now, substitution of eqn. (13) into eqn. (9) yields

$$A^6 + \frac{d}{2}A^4 + \frac{1}{4}\left(\frac{d^2}{4} - f\right)A^2 - \frac{e^2}{64} = 0, \tag{15}$$

which is cubic in A^2. Finally, substitution of eqn. (14) into eqn. (10) also results in eqn. (15).

At this point then, the four roots of eqn. (3) have been obtained. Namely, let A^2 be a real root of eqn. (15), and calculate A as the principal square root. Note that A may be real and positive (if $A^2 > 0$), or A may be purely imaginary (if $A^2 < 0$). Next, calculate B^2 and C^2 via eqns. (13) and (14). If A is real, then B^2 and C^2 are real, and if A is purely imaginary, then B^2 and C^2 are complex conjugates. In any case, find B and C as the principal square roots of B^2 and C^2 (if B^2 and C^2 are real), or as the first complex square roots of B^2 and C^2 (if B^2 and C^2 are complex). So, B and C may be real, purely imaginary or complex. The four roots of eqn. (3) are then given by eqns. (6). Finally, the four roots of eqn. (1) follow from eqn. (2), *viz.*,

$$x = -\frac{1}{4}p + A \pm iB, \qquad x = -\frac{1}{4}p - A \pm iC. \tag{16}$$