Example #1. For $y = f(x) = \dfrac{2x+3}{4x+5}$, calculate $y = f^{-1}(x)$.

SOLUTION: Switch x and $y \Rightarrow x = \dfrac{2y+3}{4y+5}$. Solve for y...

$x(4y+5) = 2y+3$, $4xy + 5x = 2y+3$, $4xy - 2y = 3-5x$,

$y(4x-2) = 3-5x$, $y = f^{-1}(x) = \dfrac{3-5x}{4x-2}$

Example #2. For $y = f(x) = \dfrac{2x+3}{4x+5}$ and $y = f^{-1}(x) = \dfrac{3-5x}{4x-2}$, verify that

(a) $f(f^{-1}(x)) = x$ (b) $f^{-1}(f(x)) = x$

SOLUTION:

(a) $$f(f^{-1}(x)) = f(f^{-1}) = \frac{2f^{-1}+3}{4f^{-1}+5} = \frac{2\left(\frac{3-5x}{4x-2}\right) + 3\left(\frac{4x-2}{4x-2}\right)}{4\left(\frac{3-5x}{4x-2}\right) + 5\left(\frac{4x-2}{4x-2}\right)} =$$

$$= \frac{\dfrac{2(3-5x)+3(4x-2)}{\cancel{4x-2}}}{\dfrac{4(3-5x)+5(4x-2)}{\cancel{4x-2}}} = \frac{2(3-5x)+3(4x-2)}{4(3-5x)+5(4x-2)} = \frac{6-10x+12x-6}{12-20x+20x-10} = \frac{2x}{2} = x$$

(b) $$f^{-1}(f(x)) = f^{-1}(f) = \frac{3-5f}{4f-2} = \frac{3\left(\frac{4x+5}{4x+5}\right) - 5\left(\frac{2x+3}{4x+5}\right)}{4\left(\frac{2x+3}{4x+5}\right) - 2\left(\frac{4x+5}{4x+5}\right)} =$$

$$= \frac{\dfrac{3(4x+5)-5(2x+3)}{\cancel{4x+5}}}{\dfrac{4(2x+3)-2(4x+5)}{\cancel{4x+5}}} = \frac{3(4x+5)-5(2x+3)}{4(2x+3)-2(4x+5)} = \frac{12x+15-10x-15}{8x+12-8x-10} = \frac{2x}{2} = x$$