3A.8. Exponential Growth and Decay

Exponential Growth

e.g. population growth

$p = p(t) \equiv$ population

$t \equiv$ time

$p_0 = p(0)$

$$\boxed{p = p_0 e^{kt}}$$

$k \equiv$ rate constant $(k > 0)$

Doubling Time (T)

$2p_0 = p_0 e^{kT}$, $2 = e^{kT}$,

$\ln 2 = \ln e^{kT}$, $kT = \ln 2$,

$$\boxed{T = \frac{\ln 2}{k}}$$

Exponential Decay

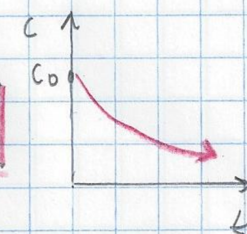
e.g. radioactive decay

$c = c(t) \equiv$ isotope concentration

$t \equiv$ time

$c_0 = c(0)$

$$\boxed{c = c_0 e^{-kt}}$$



$k \equiv$ rate constant $(k > 0)$

Half-Life (T)

$\frac{1}{2} c_0 = c_0 e^{-kT}$, $\frac{1}{2} = e^{-kT} = \frac{1}{e^{kT}}$,

$2 = e^{kT}$

$$\boxed{T = \frac{\ln 2}{k}}$$

Example #1. In 2000, the population of the United States was 282.2 million people. In 2020 it was 329.5 million. Estimate in what year the population will be 450 million.

SOLUTION:

$p = p_0 e^{kt}$, $p = 282.2\, e^{kt}$ $(t = 0 \to 2000,\ t = 20 \to 2020)$

$329.5 = 282.2\, e^{k(20)}$ $\quad e^{20k} = \frac{329.5}{282.2}$, $20k = \ln\left(\frac{329.5}{282.2}\right)$

$k = \frac{1}{20} \ln\left(\frac{329.5}{282.2}\right) = 0.007748\ldots$ /yr, $450 = 282.2\, e^{kt}$

$e^{kt} = \frac{450}{282.2}$, $kt = \ln\left(\frac{450}{282.2}\right)$, $t = \frac{1}{k} \ln\left(\frac{450}{282.2}\right) = 60.2 + 2000 = 2060$ ←

Example #2. The half-life of Uranium-235 is 0.7 billion years. After how many years will 60% of a sample of Uranium-235 have decayed?

SOLUTION:

$T = \frac{\ln 2}{k}$, $k = \frac{\ln 2}{T} = \frac{\ln 2}{0.7} = 0.990210\ldots$ / billion yrs

$C = C_0 e^{-kt}$, 60% decayed $\rightarrow$ 40% present

$0.4C_0 = C_0 e^{-kt}$, $0.4 = e^{-kt}$, $-kt = \ln(0.4)$, $t = -\frac{1}{k}\ln(0.4) = 0.9253$ billion yrs