2.1. Definition of Limits

Limits from the Left

$\lim_{x \to a^-} f(x) \equiv$ the limit as x approaches a from the left of $f(x)$

It means $\lim_{x \to a^-} f(x) \equiv f(a-\varepsilon)$, where

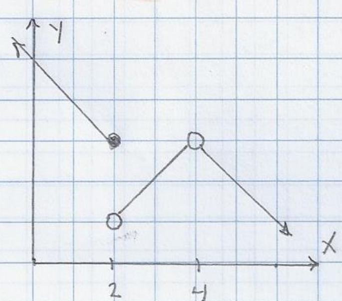
$\varepsilon \equiv$ an arbitrarily small positive number
$\equiv$ a vanishingly small " "
$\equiv$ an infinitesimally small " "

Limits from the Right

$\lim_{x \to a^+} f(x) \equiv$ the limit as x approaches a from the right of $f(x)$

It means $\lim_{x \to a^+} f(x) \equiv f(a+\varepsilon)$, where ε is as described above.

Example #1. For $y = f(x)$ as shown, calculate $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$



for

(a) $a=2$ (b) $a=4$

SOLUTION:

(a) $\lim_{x \to 2^-} f(x) = 3$ $\lim_{x \to 2^+} f(x) = 1$

(b) $\lim_{x \to 4^-} f(x) = 3$ $\lim_{x \to 4^+} f(x) = 3$

The Limit

$\lim_{x \to a} f(x) \equiv$ the limit as x approaches a of $f(x)$

If $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$, where L is some number (including ∞ and $-\infty$), then $\lim_{x \to a} f(x) = L$. Otherwise $\lim_{x \to a} f(x)$ does not exist.

Example #2. For $y = f(x)$ as in Example #1, calculate

(a) $\lim_{x\to 2} f(x)$ (b) $\lim_{x\to 4} f(x)$

SOLUTION:

(a) $\lim_{x\to 2} f(x)$ does not exist (D.N.E.) because $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$ ◄

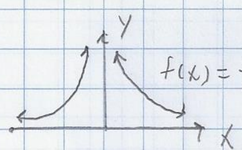
(b) $\lim_{x\to 4} f(x) = 3$ because $\lim_{x\to 4^-} f(x) = 3 = \lim_{x\to 4^+} f(x)$ ◄ Even though $f(4)$ is not defined.

Example #3 Calculate $\lim_{x\to 0} f(x)$ for

(a) $f(x) = \frac{1}{x^2}$ (b) $f(x) = \frac{1}{x}$

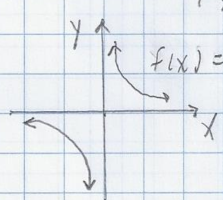
SOLUTION:

(a)



$\lim_{x\to 0^-} f(x) = \infty$, $\lim_{x\to 0^+} f(x) = \infty \Rightarrow \lim_{x\to 0} f(x) = \infty$ ◄

(b)



$\lim_{x\to 0^-} f(x) = -\infty$, $\lim_{x\to 0^+} f(x) = \infty \Rightarrow \lim_{x\to 0} f(x)$ D.N.E. ◄

Example #4. Evaluate $\lim_{x\to 0} f(x)$ for

(a) $f(x) = x\cos x$ (b) $f(x) = x\cos\left(\frac{1}{x}\right)$ (c) $f(x) = \ln|x|$

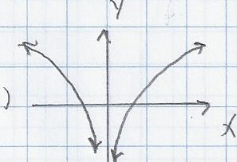
SOLUTION:

(a) $\lim_{x\to 0} x\cos x = 0\cdot\cos(0) = 0\cdot 1 = 0$ ◄

bounded between −1 and 1

(b) $\lim_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0\cdot \overbrace{\cos\left(\frac{1}{x}\right)} = 0$ ◄

(c)



$\lim_{x\to 0} \ln|x| = -\infty$ ◄