2.2. Limits Involving Infinity

Example #1: Calculate $\lim_{x\to\infty} f(x)$ for

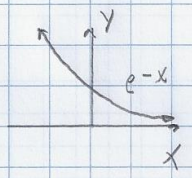
(a) $f(x)=\frac{1}{x}$ (b) $f(x)=e^{-x}$ (c) $f(x)=e^{x}$

(d) $f(x)=\sin x$ (e) $f(x)=\frac{\sin x}{x}$

SOLUTION:

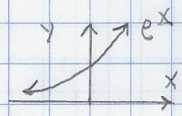
(a) $\lim_{x\to\infty}\frac{1}{x}=\frac{1}{\infty}=0$ ◀

(b)



$\lim_{x\to\infty} e^{-x}=0$ ◀

(c)



$\lim_{x\to\infty} e^{x}=\infty$ ◀

(d) $\lim_{x\to\infty}\sin x$ D.N.E. ◀ because $\sin x$ keeps oscillating between -1 & 1.

(e) $\lim_{x\to\infty}\frac{1}{x}\cdot\sin x=\frac{1}{\infty}\cdot\left[\begin{array}{c}\text{bounded}\\ \text{between}\\ -1 \text{ \& } 1\end{array}\right]=0\cdot[\text{bounded}]=0$ ◀

Horizontal Asymptotes (H.A.'s)

If $\lim_{x\to-\infty} f(x)=L$ and/or $\lim_{x\to\infty} f(x)=L$, where $L\neq\pm\infty$, then

$y=L$ is a horizontal asymptote of $y=f(x)$.

Example #2: Find the horizontal asymptotes of

(a) $f(x)=\frac{3x^2+9x}{x^2+4}$ (b) $f(x)=\frac{3x+9}{x^2+4}$

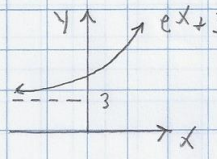
(c) $f(x)=e^{x}+3$ (d) $f(x)=e^{-x}+3$

SOLUTION:

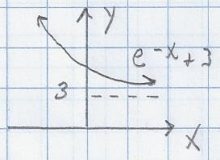
Dominates for large x

(a) $\lim_{x\to\pm\infty}\frac{3x^2+9x}{x^2+4}=\lim_{x\to\pm\infty}\frac{3x^2}{x^2}=3\Rightarrow y=3$ is H.A. ◀

(b) $\lim_{x\to\pm\infty}\frac{3x+9}{x^2+4}=\lim_{x\to\pm\infty}\frac{3x}{x^2}=\lim_{x\to\pm\infty}\frac{3}{x}=\frac{3}{\pm\infty}=0\Rightarrow y=0$ is H.A. ◀

(c) 

$\lim_{x\to-\infty}(e^x+3) = 0+3 = 3 \Rightarrow y=3$ is H.A. ◄

(d) 

$\lim_{x\to\infty}(e^{-x}+3) = 0+3 = 3 \Rightarrow y=3$ is H.A. ◄

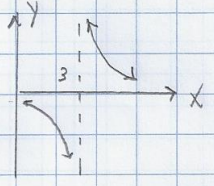
Vertical Asymptotes (V.A.'s)

If $\lim_{x\to a^-} f(x) = \pm\infty$ and/or $\lim_{x\to a^+} f(x) = \pm\infty$, then $x=a$ is a vertical asymptote of $y=f(x)$.

Example #3. Find the vertical asymptotes of

(a) $f(x) = \dfrac{x+2}{x-3}$ (b) $f(x) = \dfrac{x^2-x-6}{x^2-5x+4}$ (c) $f(x) = \ln(x-3)$

SOLUTION:

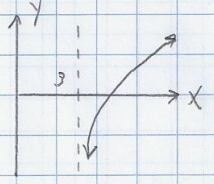
(a) 

The V.A. is where the denominator is zero $\Rightarrow$

$x=3$ is V.A. ◄

(b) $f(x) = \dfrac{(x-3)(x+2)}{(x-1)(x-4)}$ The denominator is zero $\Rightarrow$

$x=1$ and $x=4$ are V.A.'s ◄

(c) 

$x=3$ is V.A. ◄

2.2. Limits Involving Infinity

CLASS WORK

(1) Find $\lim_{x\to\infty} f(x)$ for

(a) $f(x)=\frac{1}{x^2}$ (b) $f(x)=5^x$ (c) $f(x)=5^{-x}$

(d) $f(x)=\cos x$ (e) $f(x)=\frac{\cos x}{x^2}$

(2) Find the horizontal asymptotes of

(a) $f(x)=\frac{7x^3+9x^2}{x^3-27}$ (b) $f(x)=\frac{7x^3+9x^2}{x^4-27x}$ (c) $f(x)=5^x+27$

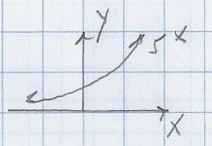
(3) Find the vertical asymptotes of

(a) $f(x)=\frac{x+9}{2x+3}$ (b) $f(x)=\frac{x^2+4x-45}{2x^2-5x-12}$ (c) $f(x)=\ln(x^2-9)$

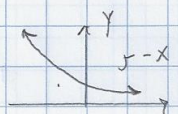
SOLUTIONS

(1)

(a) $\lim_{x\to\infty}\frac{1}{x^2}=\frac{1}{\infty}=0$ ◄

(b) 

$\lim_{x\to\infty}5^x=\infty$ ◄

(c) 

$\lim_{x\to\infty}5^{-x}=0$ ◄

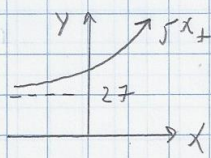
(d) $\lim_{x\to\infty}\cos x$ D.N.E. ◄ because $\cos x$ keeps oscillating between -1 & 1.

(e) $\lim_{x\to\infty}\frac{1}{x^2}\cdot\cos x=\frac{1}{\infty}\cdot\left[\begin{array}{c}\text{bounded}\\ \text{between}\\ -1 \text{ \& } 1\end{array}\right]=0\cdot[\text{bounded}]=0$ ◄

(2)

(a) $\lim_{x\to\pm\infty}\frac{7x^3+9x^2}{x^3-27}$ Dominates $=\lim_{x\to\pm\infty}\frac{7x^3}{x^3}=7\Rightarrow y=7$ is H.A. ◄

(b) $\lim_{x\to\pm\infty}\frac{7x^3+9x^2}{x^4-27x}=\lim_{x\to\pm\infty}\frac{7x^3}{x^4}=\lim_{x\to\pm\infty}\frac{7}{x}=\frac{7}{\pm\infty}=0\Rightarrow y=0$ is H.A. ◄

(c) 

$\lim_{x\to-\infty}(5^x+27)=0+27=27\Rightarrow y=27$ is H.A. ◄

2.2. Limits Involving Infinity

(3)

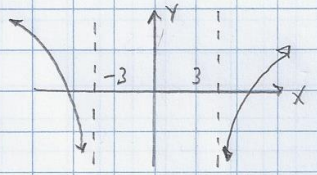
(a) $f(x) = \dfrac{x+9}{2x+3}$, $2x+3=0$, $x = -\frac{3}{2} = -1.5$ is V.A.

(b) $f(x) = \dfrac{x^2+4x-45}{2x^2-5x-12} = \dfrac{(x-5)(x+9)}{(2x+3)(x-4)} \Rightarrow x=-1.5,\ x=4$ are V.A.

$ac = (2)(-12) = -24 = -2^3 \cdot 3 = tu$, $b = -5 = t+u$, $t=-8$, $u=3$

$2x^2-8x+3x-12 = 2x(x-4)+3(x-4) = (2x+3)(x-4)$

(c)



$x=-3$ and $x=3$ are V.A.