2.2. Relative Sizes of e^x, x^2 and $\ln x$

■ $f(x) = \dfrac{x^2}{e^x}$

x	$f(x)$
10^1	4.54×10^{-3}
10^2	3.72×10^{-40}

$$\Rightarrow \lim_{x\to\infty} \frac{x^2}{e^x} = \frac{\infty}{\infty} = \frac{\text{small}}{\text{big}} = 0$$

Dominates

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \frac{\infty}{\infty} = \frac{\text{big}}{\text{small}} = \infty$$

■ $g(x) = \dfrac{\ln x}{x^2}$

x	$g(x)$
10^1	2.30×10^{-2}
10^2	4.61×10^{-4}
10^8	1.84×10^{-15}

$$\Rightarrow \lim_{x\to\infty} \frac{\ln x}{x^2} = \frac{\infty}{\infty} = \frac{\text{small}}{\text{big}} = 0$$

Dominates

$$\lim_{x\to\infty} \frac{x^2}{\ln x} = \frac{\infty}{\infty} = \frac{\text{big}}{\text{small}} = \infty$$

RULES

As $x \to \infty$

(1) Exponential functions b^x $(b>0)$ approach ∞ faster than . . .

(2) Power functions x^p $(p>0)$ approach ∞ faster than . . .

(3) Logarithmic functions $\log_b x$ $(b>0)$ approach ∞.

Example #1. Find $\lim_{x\to\infty} f(x)$ for

(a) $f(x) = \dfrac{2^x}{\log_2 x}$ (b) $f(x) = \dfrac{\log_4 x}{\sqrt{x}}$

SOLUTION:

Dominates

(a) $$\lim_{x\to\infty} \frac{2^x}{\log_2 x} = \frac{\infty}{\infty} = \frac{\text{big}}{\text{small}} = \infty$$

(b) $$\lim_{x\to\infty} \frac{\log_4 x}{\sqrt{x}} = \frac{\infty}{\infty} = \frac{\text{small}}{\text{big}} = 0$$

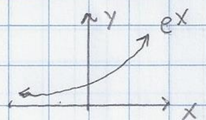
Dominates

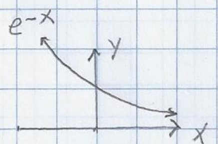
Example #2. Find $\lim_{x\to\infty} f(x)$ for.

(a) $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ (b) $f(x) = \dfrac{\ln x - 2^x}{x - 3^x}$

SOLUTION:

(a)





Dominates

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to\infty} \frac{e^x}{e^x} = 1 \blacktriangleleft$$

Dominates

Dominates

(b) $$\lim_{x\to\infty} \frac{\ln x - 2^x}{x - 3^x} = \lim_{x\to\infty} \frac{-2^x}{-3^x} = \lim_{x\to\infty} \frac{2^x}{3^x} = \frac{\infty}{\infty} = \frac{\text{small}}{\text{big}} = 0 \blacktriangleleft$$

Dominates

Example #3. For

(a) $f(x) = x^{100} - e^x$ (b) $f(x) = e^{-x} - x^2$

calculate $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$, the left end-behavior function $E_L(x)$, and the right end-behavior function $E_R(x)$.

SOLUTION:

(a) $\lim_{x\to\infty} (x^{100} - e^x) = \infty - \infty = -\infty \blacktriangleleft$ $\Rightarrow E_R(x) = -e^x \blacktriangleleft$

Dominates

$\lim_{x\to-\infty} (x^{100} - e^x) = \infty - 0 = \infty \blacktriangleleft$ $\Rightarrow E_L(x) = x^{100} \blacktriangleleft$

(b) $\lim_{x\to\infty} (e^{-x} - x^2) = 0 - \infty = -\infty \blacktriangleleft$ $\Rightarrow E_R(x) = -x^2 \blacktriangleleft$

Dominates

$\lim_{x\to-\infty} (e^{-x} - x^2) = \infty - \infty = \infty \blacktriangleleft$ $\Rightarrow E_L(x) = e^{-x} \blacktriangleleft$

2.2. Relative Sizes of e^x, x^2 and $\ln x$

CLASS WORK

(1) Find $\lim_{x\to\infty} f(x)$ for

(a) $f(x) = \dfrac{\log x}{2^x}$ (b) $f(x) = \dfrac{\sqrt{x}}{\log x}$

(c) $f(x) = \dfrac{\sqrt{x^2+8x+3}}{x+\log_7 x}$ (d) $f(x) = \dfrac{2^x - \sqrt{x}}{x^{10}+\ln x}$

(2) For

(a) $f(x) = \sqrt{|x|} - e^{-x}$ (b) $f(x) = 3^x - x^2$

calculate $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$, the right end-behavior function $E_R(x)$, and the left end-behavior function $E_L(x)$.

SOLUTIONS

(1)

(a) $\lim_{x\to\infty} \dfrac{\log x}{2^x} = \dfrac{\infty}{\infty} = \dfrac{\text{small}}{\text{big}} = 0$

Dominates

(b) $\lim_{x\to\infty} \dfrac{\sqrt{x}}{\log x} = \dfrac{\infty}{\infty} = \dfrac{\text{big}}{\text{small}} = \infty$

Dominates

(c) $\lim_{x\to\infty} \dfrac{\sqrt{x^2+8x+3}}{x+\log_7 x} = \lim_{x\to\infty} \dfrac{\sqrt{x^2}}{x} = \lim_{x\to\infty} \dfrac{x}{x} = 1$

(d) $\lim_{x\to\infty} \dfrac{2^x - \sqrt{x}}{x^{10}+\ln x} = \lim_{x\to\infty} \dfrac{2^x}{x^{10}} = \dfrac{\infty}{\infty} = \dfrac{\text{big}}{\text{small}} = \infty$

Dominates

2.2. Relative Sizes of e^x, x^2 and $\ln x$

(2)

(a) $\lim_{x\to\infty}\left(\sqrt{|x|}-e^{-x}\right)=\infty-0=\infty \Rightarrow E_R(x)=\sqrt{x}$

Dominates

$\lim_{x\to-\infty}\left(\sqrt{|x|}-e^{-x}\right)=\infty-\infty=-\infty \Rightarrow E_L(x)=-e^{-x}$

(b) $\lim_{x\to\infty}\left(3^x-x^2\right)=\infty-\infty=\infty \Rightarrow E_R(x)=3^x$

Dominates

$\lim_{x\to-\infty}\left[3^x-x^2\right]=0-\infty=-\infty \Rightarrow E_L(x)=-x^2$