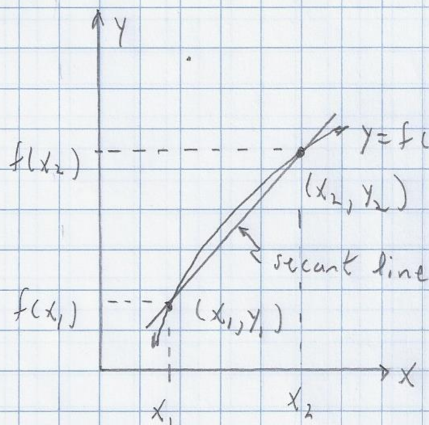
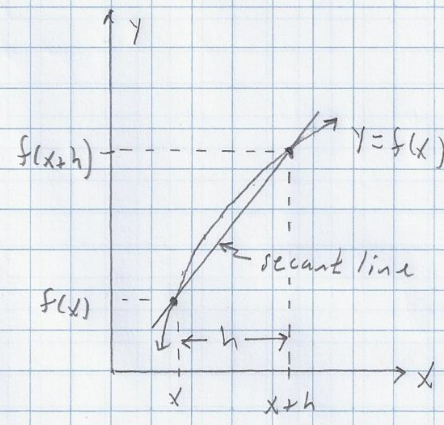
2.4. Average and Instantaneous Rates of Change

Average Rate of Change





$m_s \equiv$ slope of the secant line
$\equiv$ average rate of change in f with respect to x on $[x_1, x_2]$
$\equiv$ " " " " " " " " " " " $[x, x+h]$

$$m_s = \frac{\text{rise}}{\text{run}} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x+h) - f(x)}{h}$$

Example #1. In 1920, the yearly U.S. oil production was 13.2 million barrels. In 2020, it was 153.8 million barrels. Find the average rate of change in U.S. oil production over the time interval.

SOLUTION:

$(x_1, y_1) = (1920, 13.2) \qquad (x_2, y_2) = (2020, 153.8)$

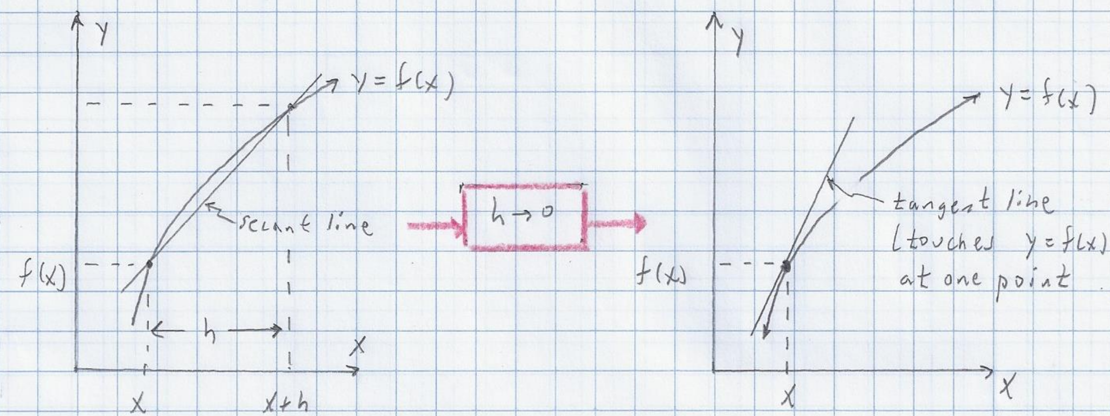
$$m_s = \frac{y_2 - y_1}{x_2 - x_1} = \frac{153.8 - 13.2}{2020 - 1920} = \frac{140.6}{100} = 1.406 \ \frac{\text{million barrels}}{\text{year}}$$

Example #2. Calculate the average rate of change of f with respect to x over $x \in [3, 8]$ for $f(x) = 5x^2 + 3x - 2$.

SOLUTION:

$$m_s = \frac{f(8) - f(3)}{8 - 3} = \frac{342 - 52}{8 - 3} = \frac{290}{5} = 58$$

Instantaneous Rate of Change



$m_T \equiv$ slope of the tangent line
$\equiv$ instantaneous rate of change of f with respect to x at x
$\equiv$ slope of the function at x.

$$m_T = \lim_{h\to 0} m_s = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Example # 3. For $y = f(x) = \frac{1}{3}x^3 - 4x^2 + 12x - 6$,

(a) Find m_T at $x = 3$
(b) Find the equation of the line tangent to $y = f(x)$ at $x = 3$
(c) Find the equation of the line normal to (or perpendicular to) $y = f(x)$ at $x = 3$
(d) Graph $y = f(x)$ and the tangent and normal lines with your calculator. Window: $x \in [-8, 10]$, $y \in [-6, 6]$

SOLUTION:

(a) $f(3) = 3$ $\quad m_T = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$

1
1 1
1 2 1
1 3 3 1

$f(3+h) = \frac{1}{3}(3+h)^3 - 4(3+h)^2 + 12(3+h) - 6$

$(3+h)^3 = 3^3 + 3\cdot 3^2 h + 3\cdot 3^1 \cdot h^2 + h^3 = 27 + 27h + 9h^2 + h^3$

$f(3+h) = \frac{1}{3}(27 + 27h + 9h^2 + h^3) - 4(9 + 6h + h^2) + 12(3+h) - 6 =$

$= 9 + 9h + 3h^2 + \frac{1}{3}h^3 - 36 - 24h - 4h^2 + 36 + 12h - 6 =$

$= 3 - 3h - h^2 + \frac{1}{3}h^3$

$f(3+h) - f(3) = -3h - h^2 + \frac{1}{3}h^3$ $\frac{f(3+h) - f(3)}{h} = -3 - h + \frac{1}{3}h^2$

$m_T = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0}\left(-3 - h + \frac{1}{3}h^2\right) = -3$ ←

(b) $y = -3x + b$ $3 = -3(3) + b$ $b = 12$ $\Rightarrow$ $y = -3x + 12$ ←

(c) The slope of the normal line is the opposite reciprocal

$y = \frac{1}{3}x + b$ $3 = \frac{1}{3}(3) + b$ $b = 2$ $\Rightarrow$ $y = \frac{1}{3}x + 2$ ←

(d)

```
Plot1 Plot2 Plot3
\Y1=(1/3)*X^3-4X
^2+12X-6
\Y2=-3X+12
\Y3=X/3+2
\Y4=
\Y5=
\Y6=
```

```
WINDOW
 Xmin=-8
 Xmax=10
 Xscl=2
 Ymin=-6
 Ymax=6
 Yscl=1
↓Xres=■
```