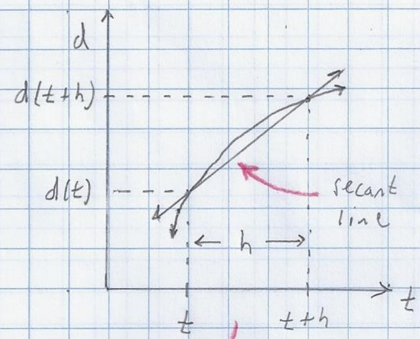
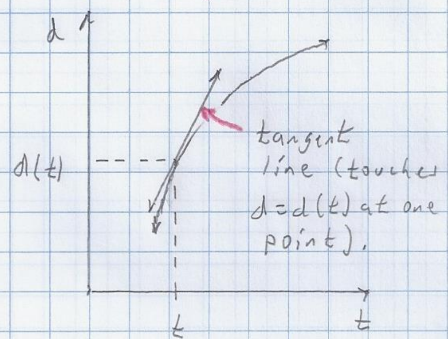
2.4. Velocities



$\boxed{h \to 0}$



$d \equiv$ distance

$t \equiv$ time

$m_s(t,h) \equiv$ slope of the secant line

$\equiv$ average velocity on $[t, t+h]$

$$\boxed{m_s(t,h) = \frac{d(t+h) - d(t)}{h}} \quad \boxed{= \frac{\Delta d}{\Delta t}}$$

$m_t(t) \equiv$ slope of the tangent line

$\equiv$ (instantaneous) velocity at t

$v \equiv$ velocity

$$\boxed{v = \lim_{h \to 0} m_s(t,h) = \lim_{h \to 0} \frac{d(t+h) - d(t)}{h}}$$

Example. A car completes a 0.25-mile (1320 ft) drag race in 6 seconds. The distance d (in feet) of the car as a function of time t (in seconds) is given by

$$d = \frac{55}{27}\left(24t^2 - t^3\right).$$

Find,

(a) The average velocity of the car (in both $\frac{ft}{sec}$ and $\frac{mi}{hr}$) on $t \in [2,4]$ sec.

(b) The velocity $v = v(t)$ of the car.

(c) The maximum velocity of the car during the drag race (in both $\frac{ft}{sec}$ and $\frac{mi}{hr}$).

(d) The average velocity of the car over the drag race (in both $\frac{ft}{sec}$ and $\frac{mi}{hr}$).

SOLUTION:

2.4. Velocities

(a) $$\frac{d(4)-d(2)}{4-2} = \frac{\frac{17,600}{27} - \frac{4840}{27}}{2} = \frac{12,760}{54} = \frac{6380}{27} = 263.30 \frac{ft}{sec}$$

$$\frac{263.30 \cancel{ft}}{\cancel{sec}} \left(\frac{1 mi}{5280 \cancel{ft}}\right)\left(\frac{3600 \cancel{sec}}{1 hr}\right) = 161.11 \frac{mi}{hr}$$

(b) $$v(t) = \lim_{h \to 0} \frac{d(t+h)-d(t)}{h}$$

$$d(t+h) = \frac{55}{27}\left[24(t+h)^2 - (t+h)^3\right] = \frac{55}{27}\left[24(t^2+2th+h^2) - (t^3+3t^2h+3th^2+h^3)\right]$$

$$= \frac{55}{27}\left[24t^2+48th+24h^2-t^3-3t^2h-3th^2-h^3\right] =$$

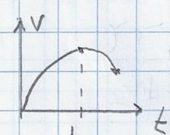
$$= \underbrace{\frac{55}{27}\left[24t^2-t^3\right]}_{d(t)} + \frac{55}{27}\left[48th+24h^2-3t^2h-3th^2-h^3\right]$$

$$d(t+h)-d(t) = \frac{55}{27}\left(48th+24h^2-3t^2h-3th^2-h^3\right)$$

$$\frac{d(t+h)-d(t)}{h} = \frac{55}{27}\left(48t+24h-3t^2-3th-h^2\right)$$

$$v(t) = \frac{55}{27}\left(48t-3t^2\right) = \frac{55}{9}\left(16t-t^2\right) = -\frac{55}{9}t^2 + \frac{880}{9}t$$

(c)



$$t_{max} = -\frac{b}{2a} = -\frac{\frac{880}{9}}{2\left(-\frac{55}{9}\right)} = \frac{880}{110} = 8 \text{ sec}$$ which is after the drag race ends ⇒ V is increasing ⇒

$$v_{max} = v(6) = \frac{1100}{3} = 366\tfrac{2}{3} \frac{ft}{sec}$$

$$\frac{366\frac{2}{3} \cancel{ft}}{\cancel{sec}} \left(\frac{1 mi}{5280 \cancel{ft}}\right)\left(\frac{3600 \cancel{sec}}{1 hr}\right) = 250 \frac{mi}{hr}$$

(d) $$v_{ave} = \frac{d(6)-d(0)}{6-0} = \frac{1320-0}{6-0} = 220 \frac{ft}{sec}$$

$$\frac{220 \cancel{ft}}{\cancel{sec}} \left(\frac{1 mi}{5280 \cancel{ft}}\right)\left(\frac{3600 \cancel{sec}}{1 hr}\right) = 150 \frac{mi}{hr}$$

CLASS WORK

A car completes a 0.25-mile (1320 ft) drag race in 8 seconds. The distance d (in feet) as a function of time t (in seconds) is given by

$$d = \frac{33}{32}\left(28t^2 - t^3\right).$$

Find,

(a) The average velocity of the car (in both $\frac{ft}{sec}$ and $\frac{mi}{hr}$) on $t \in [3,5]$ sec.

(b) The velocity $v = v(t)$ of the car.

(c) The maximum velocity of the car during the drag race (in both $\frac{ft}{sec}$ and $\frac{mi}{hr}$).

(d) The average velocity of the car over the drag race (in both $\frac{ft}{sec}$ and $\frac{mi}{hr}$).

SOLUTION

(a) $$\frac{d(5)-d(3)}{5-3} = \frac{\frac{18,975}{32} - \frac{7425}{32}}{2} = \frac{11,550}{64} = \frac{5775}{32} = 180\frac{15}{32}\ \frac{ft}{sec}$$

$$\frac{180\frac{15}{32}\ \cancel{ft}}{\cancel{sec}}\left(\frac{1\ mi}{5280\ \cancel{ft}}\right)\left(\frac{3600\ \cancel{sec}}{1\ hr}\right) = 123\frac{3}{64}\ \frac{mi}{hr}$$

(b) $$d(t+h) = \frac{33}{32}\left[28(t+h)^2 - (t+h)^3\right] = \frac{33}{32}\left[28(t^2+2th+h^2) - (t^3+3t^2h+3th^2+h^3)\right] =$$

$$= \frac{33}{32}\left(28t^2 + 56th + 28h^2 - t^3 - 3t^2h - 3th^2 - h^3\right) =$$

$$= \underbrace{\frac{33}{32}\left(28t^2 - t^3\right)}_{d(t)} + \frac{33}{32}\left(56th + 28h^2 - 3t^2h - 3th^2 - h^3\right)$$

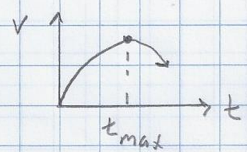
$$d(t+h) - d(t) = \frac{33}{32}\left(56th + 28h^2 - 3t^2h - 3th^2 - h^3\right)$$

$$\frac{d(t+h)-d(t)}{h} = \frac{33}{32}\left(56t + 28h - 3t^2 - 3th - h^2\right)$$

$$v = \frac{33}{32}\left(56t - 3t^2\right) = -\frac{99}{32}t^2 + \frac{231}{4}t$$

2.4. velocities

(c)



$$t_{max} = -\frac{b}{2a} = \cancel{-}\frac{\frac{231}{4}}{2\left(\cancel{-}\frac{99}{32}\right)} = \frac{\frac{231}{4}}{\frac{99}{16}} = \frac{231}{4}\cdot\frac{16}{99} = \frac{231\cdot 4}{99} = 9\tfrac{1}{3}\,\text{sec}$$

which is after the drag race is over $\Rightarrow$ v is increasing $\Rightarrow$

$$v_{max} = v(8) = 264\,\frac{ft}{sec} \blacktriangleleft \qquad \frac{264\,\cancel{ft}}{\cancel{sec}}\left(\frac{1\,mi}{5280\,\cancel{ft}}\right)\left(\frac{3600\,\cancel{sec}}{1\,hr}\right) = 180\,\frac{mi}{hr} \blacktriangleleft$$

(d)

$$v_{ave} = \frac{d(8)-d(0)}{8-0} = \frac{1320}{8} = 165\,\frac{ft}{sec} \blacktriangleleft$$

$$\frac{165\,\cancel{ft}}{\cancel{sec}}\left(\frac{1\,mi}{5280\,\cancel{ft}}\right)\left(\frac{3600\,\cancel{sec}}{1\,hr}\right) = 112.5\,\frac{mi}{hr} \blacktriangleleft$$