3.3. Irrational and Rational Powers

We know that the Power Rule $f(x) = x^p \Rightarrow f'(x) = px^{p-1}$ is valid if p is an integer.

Irrational Powers.

We will look at this case by way of an example.

Example #1. For $f(x) = x^{1/\pi}$

(a) Calculate $f'(x)$ via the Power Rule.

(b) Find the line tangent to $f(x)$ at $x=3$.

(c) Graph $f(x)$ and the tangent line with your calculator using the window $x \in [0, 8]$ and $y \in [0, 3]$.

SOLUTION:

(a) $f'(x) = \frac{1}{\pi} x^{\frac{1}{\pi}-1}$

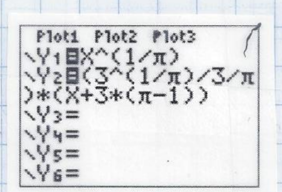
(b) $f'(3) = \frac{1}{\pi} 3^{\frac{1}{\pi}-1} = \frac{3^{\frac{1}{\pi}}}{3\pi}$ $\quad f(3) = 3^{1/\pi}$ so $(3, 3^{1/\pi})$ is on $y = f(x)$.

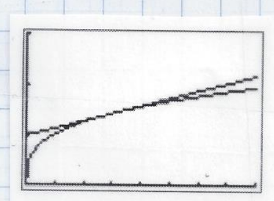
$y = \frac{3^{\frac{1}{\pi}}}{3\pi} x + b$, $\quad 3^{\frac{1}{\pi}} = \frac{3^{\frac{1}{\pi}}}{3\pi}(3) + b = \frac{3^{\frac{1}{\pi}}}{\pi} + b$, $\quad b = 3^{\frac{1}{\pi}} - \frac{3^{\frac{1}{\pi}}}{\pi} =$

$= 3^{\frac{1}{\pi}}\left[1 - \frac{1}{\pi}\right] = 3^{\frac{1}{\pi}}\left[\frac{\pi}{\pi} - \frac{1}{\pi}\right] = \frac{3^{\frac{1}{\pi}}}{\pi}(\pi - 1) = \frac{3^{\frac{1}{\pi}}}{3\pi} \cdot 3(\pi - 1)$

$y = \frac{3^{\frac{1}{\pi}}}{3\pi} x + \frac{3^{\frac{1}{\pi}}}{3\pi} \cdot 3(\pi - 1)$, $\quad y = \frac{3^{\frac{1}{\pi}}}{3\pi}\left[x + 3(\pi - 1)\right]$

(c)





The tangent line is indeed tangent to the graph $\Rightarrow$ p can be irrational.

Rational Powers

Since the Power Rule is valid for p being an integer and p being an irrational number, then it must be true for p being a rational number. In other words, the Power Rule is valid for p being any real number.

Example #2: Calculate $f'(x)$ for

(a) $f(x) = \sqrt{x}$ (b) $f(x) = \sqrt[3]{x^8}$ (c) $f(x) = \dfrac{1}{\sqrt[5]{x^3}}$

SOLUTION:

(a) $f(x) = \sqrt{x} = x^{1/2}$, $f'(x) = \frac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$ ◄

(b) $f(x) = \sqrt[3]{x^8} = x^{8/3}$, $f'(x) = \frac{8}{3}x^{\frac{8}{3}-1} = \frac{8}{3}x^{5/3} = \dfrac{8\cdot\sqrt[3]{x^5}}{3}$ ◄

(c) $f(x) = \dfrac{1}{\sqrt[5]{x^3}} = \dfrac{1}{x^{3/5}} = x^{-3/5}$, $f'(x) = -\frac{3}{5}x^{-3/5-1} = -\frac{3}{5}x^{-8/5} =$

$= -\dfrac{3}{5x^{8/5}} = -\dfrac{3}{5\cdot\sqrt[5]{x^8}}$ ◄

CLASS WORK

1) Calculate $f'(x)$ for

(a) $f(x) = \sqrt[3]{x}$ (b) $f(x) = \sqrt[7]{x^3}$ (c) $f(x) = \dfrac{1}{\sqrt[4]{x^5}}$

2) For $f(x) = \sqrt{x} \Rightarrow f'(x) = \dfrac{1}{2\sqrt{x}}$, find the equation of the line tangent to $y = f(x)$ at $x = 4$. Graph $f(x)$ and the tangent line with your calculator using the window $x \in [0,10]$ and $y \in [0,4]$. Show me the graph.

SOLUTIONS

1)
a) $f(x) = \sqrt[3]{x} = x^{1/3}$, $f'(x) = \frac{1}{3}x^{1/3-1} = \frac{1}{3}x^{-2/3} = \dfrac{1}{3\cdot\sqrt[3]{x^2}}$ ◄

b) $f(x) = \sqrt[7]{x^3} = x^{3/7}$, $f'(x) = \frac{3}{7}x^{\frac{3}{7}-1} = \frac{3}{7}x^{-\frac{4}{7}} = \dfrac{3}{7\cdot\sqrt[7]{x^4}}$ ◄

c) $f(x) = \dfrac{1}{\sqrt[4]{x^5}} = x^{-\frac{5}{4}}$, $f'(x) = -\frac{5}{4}x^{-\frac{5}{4}-1} = -\frac{5}{4}x^{-9/4} = -\dfrac{5}{4\cdot\sqrt[4]{x^9}}$ ◄

3.3. Irrational and Rational Powers

2) $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$; $y = \frac{1}{4}x + b$, $f(4) = \sqrt{4} = 2$ so $(4, 2)$ is on the graph.

$2 = \frac{1}{4}(4) + b$, $b = 1$, $y = \frac{1}{4}x + 1$