Rules for Differentiability at a Point

1) If $y=f(x)$ is not continuous at $x=a$, then $f'(a)$ does not exist. The derivatives from the left and the right at $x=a$ will differ.

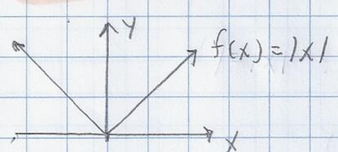
2) If $y=f(x)$ is continuous at $x=a$, but is not smooth there (e.g., the function has a kink at $x=a$), then $f'(a)$ does not exist. Once again, the derivatives from the left and the right will differ.

3) If $y=f(x)$ is continuous and smooth at $x=a$, then $f'(a)$ exists. The derivatives from the left and the right will be equal.

Conversely

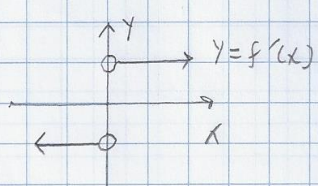
If $f'(a)$ exists, then $y=f(x)$ is continuous and smooth at $x=a$.

Example #1. Calculate and graph $f'(x)$ for $f(x)=|x|$.



SOLUTION: The derivative does not exist at $x=0$ due to the kink there.

$$f'(x) = \begin{bmatrix} -1 & ; & -\infty < x < 0 \\ 1 & ; & 0 < x < \infty \end{bmatrix} = \frac{x}{|x|} \quad (x \neq 0)$$



For Examples 2 through 4:

(a) Graph $f(x)$.

(b) State whether or not $f(x)$ is continuous at $x=4$.

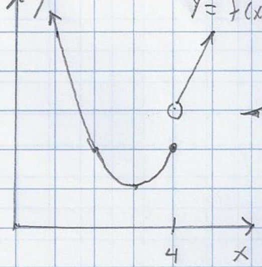
(c) Use the definitions of the derivatives from the left and from the right to calculate $f'(4)$.

(d) State if $f'(4)$ exists or not.

(e) Calculate $f'(x)$ by straightforward differentiation and graph $y=f'(x)$.

Example #2. $f(x) = \begin{bmatrix} x^2 - 6x + 10, & -\infty < x \le 4 \\ 2x - 5, & 4 < x < \infty \end{bmatrix}$

SOLUTION:

(a) 

(b) $f(x)$ is not continuous at $x = 4$

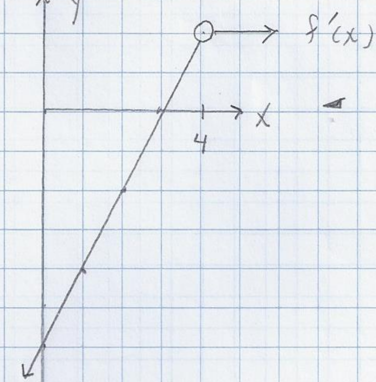
(c) $\lim\limits_{x \to 4^-} \dfrac{f(4) - f(x)}{4 - x} = \lim\limits_{x \to 4^-} \dfrac{2 - (x^2 - 6x + 10)}{4 - x} =$

$= \lim\limits_{x \to 4^-} \dfrac{-x^2 + 6x - 8}{4 - x} = \lim\limits_{x \to 4^-} \dfrac{\cancel{(4-x)}(x-2)}{\cancel{4-x}} = 4 - 2 = 2$

$\lim\limits_{x \to 4^+} \dfrac{f(x) - f(4)}{x - 4} = \lim\limits_{x \to 4^+} \dfrac{2x - 5 - 2}{x - 4} = \lim\limits_{x \to 4^+} \dfrac{2x - 7}{x - 4} = \dfrac{2(4) - 7}{4 - 4} = \dfrac{1}{0} = \infty$

(d) $f'(4)$ does not exist due to (b). Note that the left and right derivatives in (c) differ.

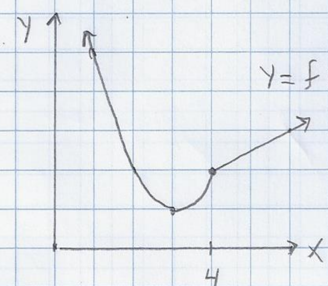
(e) $f'(x) = \begin{bmatrix} 2x - 6, & -\infty < x < 4 \\ 2, & 4 < x < \infty \end{bmatrix}$



Example #3.

$f(x) = \begin{bmatrix} x^2 - 6x + 10, & -\infty < x \le 4 \\ \frac{1}{2}x, & 4 < x < \infty \end{bmatrix}$

SOLUTION:

(a) 

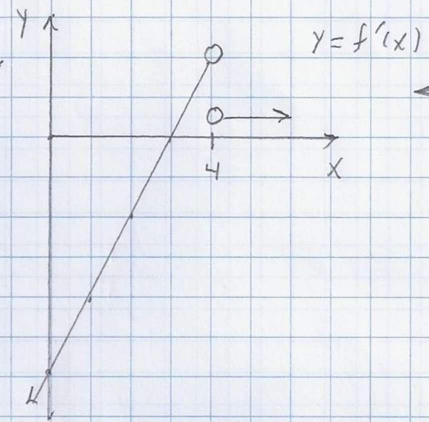
(b) $f(x)$ is continuous at $x = 4$

(c) $\lim\limits_{x \to 4^-} \dfrac{f(4) - f(x)}{4 - x} = 2$

$\lim\limits_{x \to 4^+} \dfrac{f(x) - f(4)}{x - 4} = \lim\limits_{x \to 4^+} \dfrac{\frac{1}{2}x - 2}{x - 4} = \lim\limits_{x \to 4^+} \dfrac{\frac{1}{2}\cancel{(x-4)}}{\cancel{x-4}} = \dfrac{1}{2}$

(d) $f'(4)$ does not exist due to the kink. Note that the left and right derivatives differ.

(e) $f'(x) = \begin{bmatrix} 2x-6, & -\infty < x < 4 \\ \frac{1}{2}, & 4 < x < \infty \end{bmatrix}$

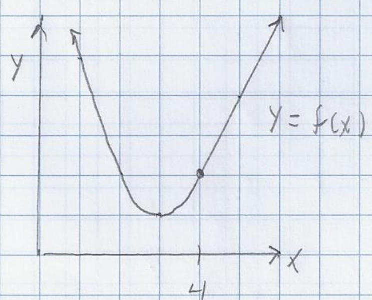


Example #4.

$f(x) = \begin{bmatrix} x^2-6x+10, & -\infty < x \le 4 \\ 2x-6, & 4 < x < \infty \end{bmatrix}$

SOLUTION:

(a)



(b) $f(x)$ is continuous at $x = 4$

(c) $\lim\limits_{x \to 4^-} \dfrac{f(4)-f(x)}{4-x} = 2$

$\lim\limits_{x \to 4^+} \dfrac{f(x)-f(4)}{x-4} = \lim\limits_{x \to 4^+} \dfrac{2x-6-2}{x-4} =$

$= \lim\limits_{x \to 4^+} \dfrac{2x-8}{x-4} = \lim\limits_{x \to 4^+} \dfrac{2\cancel{(x-4)}}{\cancel{(x-4)}} = 2$

(d) $f'(4)$ exists. In fact, $f'(4) = 2$, which is the value of the left and right derivatives.

(e) $f'(x) = \begin{bmatrix} 2x-6, & -\infty < x \le 4 \\ 2, & 4 < x < \infty \end{bmatrix}$