Notation: $f(x+h)-f(x)=\Delta f$, $h=\Delta x$, $f(x+\Delta x)-f(x)=\Delta f$

$$\frac{df}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x}.$$

Note that $\lim\limits_{\Delta x\to 0}\Delta f=\lim\limits_{\Delta x\to 0} f(x+\Delta x)-f(x)=f(x)-f(x)=0.$

Also note that $f(x+\Delta x)=f(x)+\Delta f.$

Consider

$$f(x)=u(x)v(x)\ ,\quad \frac{df}{dx}=\frac{d}{dx}\left[u(x)v(x)\right]=\lim_{\Delta x\to 0}\frac{u(x+\Delta x)v(x+\Delta x)-u(x)v(x)}{\Delta x}=$$

$$=\lim_{\Delta x\to 0}\frac{\{u(x)+\Delta u\}\{v(x)+\Delta v\}-u(x)v(x)}{\Delta x}=$$

$$=\lim_{\Delta x\to 0}\frac{\cancel{u(x)v(x)}+u(x)\Delta v+\Delta u\, v(x)+\Delta u\Delta v-\cancel{u(x)v(x)}}{\Delta x}=$$

$$=\lim_{\Delta x\to 0}\left[\frac{\Delta u}{\Delta x}v(x)+u(x)\frac{\Delta v}{\Delta x}+\frac{\Delta u}{\Delta x}\Delta v\right]=\frac{du}{dx}v(x)+u(x)\frac{dv}{dx}+\frac{du}{dx}\cdot 0 \Rightarrow$$

$$\boxed{(uv)'=u'v+uv'}$$

Product Rule

Consider

$$f(x)=\frac{u(x)}{v(x)}\ ,\quad \frac{df}{dx}=\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right]=\lim_{\Delta x\to 0}\frac{\frac{u(x+\Delta x)}{v(x+\Delta x)}-\frac{u(x)}{v(x)}}{\Delta x}=$$

$$=\lim_{\Delta x\to 0}\frac{\frac{u(x+\Delta x)}{v(x+\Delta x)}\cdot\frac{v(x)}{v(x)}-\frac{u(x)}{v(x)}\cdot\frac{v(x+\Delta x)}{v(x+\Delta x)}}{\Delta x}=\lim_{\Delta x\to 0}\frac{u(x+\Delta x)v(x)-u(x)v(x+\Delta x)}{\Delta x\, v(x+\Delta x)v(x)}=$$

$$=\lim_{\Delta x\to 0}\frac{\{u(x)+\Delta u\}v(x)-u(x)\{v(x)+\Delta v\}}{\Delta x\, v(x+\Delta x)v(x)}=$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{u(x)v(x)} + \Delta u\, v(x) - \cancel{u(x)v(x)} - u(x)\Delta v}{\Delta x\, v(x+\Delta x)\, v(x)}$$

$$= \lim_{\Delta x \to 0} \left[\frac{\frac{\Delta u}{\Delta x} v(x) - u(x)\frac{\Delta v}{\Delta x}}{v(x+\Delta x)\, v(x)} \right] = \frac{\frac{du}{dx} v(x) - u(x)\frac{dv}{dx}}{v(x)\cdot v(x)} \Rightarrow$$

$$\boxed{\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}} \leftarrow \text{Quotient Rule}$$

Example #1. Differentiate $f(x) = (2x+3)(3x+7)$ two ways, i.e., by
(a) expanding then differentiating, and
(b) using the Product Rule then expanding.

SOLUTION:

(a) $f(x) = (2x+3)(3x+7) = 6x^2 + 14x + 9x + 21 = 6x^2 + 23x + 21$

$f'(x) = 12x + 23$ ◄

(b) $f'(x) = u'v + uv' = (2)(3x+7) + (2x+3)(3) = 6x + 14 + 6x + 9 = 12x + 23$ ◄

Example #2. Differentiate $f(x) = \dfrac{15x^3 + 14x^2 - 3x - 2}{5x-2}$ two ways, i.e., by
(a) dividing then differentiating, and
(b) using the Quotient Rule then dividing.

SOLUTION:

(a)

$$\begin{array}{r} 3x^2 + 4x + 1 \\ 5x-2 \overline{\smash{)}\,15x^3 + 14x^2 - 3x - 2} \\ -15x^3 + 6x^2 \qquad\qquad \\ \hline 20x^2 - 3x - 2 \\ -20x^2 + 8x \quad\;\; \\ \hline 5x - 2 \\ 5x - 2 \\ \hline 0 \end{array}$$

$f(x) = 3x^2 + 4x + 1$

$f'(x) = 6x + 4$ ◄

(b) $$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} = \frac{(45x^2 + 28x - 3)(5x-2) - (15x^3 + 14x^2 - 3x - 2)(5)}{(5x-2)^2} =$$

$$= \frac{(225x^3 + 50x^2 - 71x + 6) - (75x^3 + 70x^2 - 15x - 10)}{(5x-2)^2} =$$

$$= \frac{150x^3 - 20x^2 - 56x + 16}{25x^2 - 20x + 4} = f'(x)$$

$$\begin{array}{r} 6x + 4 \\ 25x^2 - 20x + 4 \overline{) \, 150x^3 - 20x^2 - 56x + 16} \\ -150x^3 + 120x^2 - 24x \\ \hline 100x^2 - 80x + 16 \\ -100x^2 + 80x - 16 \\ \hline 0 \end{array}$$

$\Rightarrow f'(x) = 6x + 4$

CLASS WORK

1) a) use the Product Rule twice to show that $(abc)' = a'bc + ab'c + abc'$.
b) What do you think the Product Rule for $(abcd)'$ is?

2) Differentiate $f(x) = (2x+3)(4x+5)$ two ways, i.e., by
a) expanding then differentiating, and
b) using the product rule then expanding.

3) Differentiate $f(x) = \dfrac{6x^3 + 31x^2 + 19x - 56}{2x + 7}$ two ways, i.e., by

a) dividing and then differentiating, and
b) using the Quotient Rule then dividing.

SOLUTIONS:

1) a) $(abc)' = ([ab]c)' = [ab]'c + [ab]c' = [a'b + ab']c + [ab]c' =$
$= a'bc + ab'c + abc'$

b) $(abcd)' = a'bcd + ab'cd + abc'd + abcd'$

2)
a) $f(x) = (2x+3)(4x+5) = 8x^2 + 10x + 12x + 15 = 8x^2 + 22x + 15$,
$f'(x) = 16x + 22$

b) $f'(x) = u'v + uv' = (2)(4x+5) + (2x+3)(4) = 8x + 10 + 8x + 12 = 16x + 22$

3)

a)

$$\begin{array}{r} 3x^2 + 5x - 8 \\ 2x+7 \overline{\big)\, 6x^3 + 31x^2 + 19x - 56} \\ -6x^3 - 21x^2 \qquad\qquad\quad \\ \hline 10x^2 + 19x - 56 \\ -10x^2 - 35x \qquad \\ \hline -16x - 56 \\ 16x + 56 \\ \hline 0 \end{array}$$

$f(x) = 3x^2 + 5x - 8$

$f'(x) = 6x + 5$

b) $f'(x) = \left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2} = \dfrac{(18x^2 + 62x + 19)(2x+7) - (6x^3 + 31x^2 + 19x - 56)(2)}{(2x+7)^2} =$

$= \dfrac{(36x^3 + 250x^2 + 472x + 133) - (12x^3 + 62x^2 + 38x - 112)}{(2x+7)^2} =$

$= \dfrac{24x^3 + 188x^2 + 434x + 245}{4x^2 + 28x + 49}$

$$\begin{array}{r} 6x + 5 \qquad\qquad\quad \\ 4x^2 + 28x + 49 \overline{\big)\, 24x^3 + 188x^2 + 434x + 245} \\ -24x^3 - 168x^2 - 294x \qquad\quad \\ \hline 20x^2 + 140x + 245 \\ -20x^2 - 140x - 245 \\ \hline 0 \end{array}$$

$f'(x) = 6x + 5$