3.4. Equations of Motion

In general...

$t \equiv$ time $\quad x \equiv$ position $\quad v \equiv$ velocity $\quad a \equiv$ acceleration

$$\boxed{v = \frac{dx}{dt}} \qquad \boxed{a = \frac{dv}{dt} = \frac{d^2x}{dt^2}} \qquad \boxed{\text{speed} = |v|}$$

Motion under Gravitation

$x_0 \equiv$ position at time $t = 0$

$v_0 \equiv$ velocity at time $t = 0$

$g \equiv$ acceleration of gravity $= 32.2 \frac{ft}{s^2} = 9.81 \frac{N}{m^2}$

$$x = x_0 + v_0 t - \frac{1}{2} g t^2$$

$$v = \frac{dx}{dt} = v_0 - gt$$

$$a = -g$$

Example #1. A stock car completes a 0.25 mi (=1320 ft) drag race in 8 seconds. The position of the car as a function of time is

$$x = \frac{33}{32}\left(28t^2 - t^3\right),$$

where t is in seconds and x is in feet. Find

(a) The velocity of the car

(b) The acceleration of the car

(c) The maximum velocity of the car in both $\frac{ft}{sec}$ and $\frac{mi}{hr}$

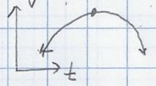
(d) The average velocity of the car during the race in both $\frac{ft}{sec}$ and $\frac{mi}{hr}$

(e) The maximum acceleration of the car.

SOLUTION:

(a) $$v = \frac{dx}{dt} = \frac{33}{32}\left(56t - 3t^2\right) = -\frac{99}{32}t^2 + \frac{231}{4}t$$

(b) $$a = \frac{dv}{dt} = -\frac{99}{16}t + \frac{231}{4}$$

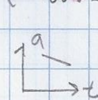
(c) $t_{max} = \dfrac{-b}{2a}$  $t_{max} = \dfrac{-\frac{231}{4}}{2\left(-\frac{99}{32}\right)} = \dfrac{\frac{231}{4}}{\frac{99}{16}} = \dfrac{231}{4} \cdot \dfrac{16}{99} = 9\frac{1}{3}\,sec$

which is out of the domain $t \in [0, 8]\,sec \Rightarrow v$ is increasing, so

$v_{max} = v(8) = 264 \frac{ft}{s}$ ◄ $\quad \dfrac{264\,ft}{sec}\left(\dfrac{1\,mi}{5280\,ft}\right)\left(\dfrac{3600\,sec}{1\,hr}\right) = 180 \dfrac{mi}{hr}$ ◄

(d) v_{ave} is the slope of the secant

$v_{ave} = \dfrac{x(8) - x(0)}{8 - 0} = \dfrac{1320}{8} = 165 \dfrac{ft}{sec}$ ◄ $\quad \dfrac{165\,ft}{sec}\left(\dfrac{1\,mi}{5280\,ft}\right)\left(\dfrac{3600\,sec}{1\,hr}\right) = 112.5 \dfrac{mi}{hr}$ ◄

(e) The slope of the acceleration is negative  $\Rightarrow$

$a_{max} = a(0) = \frac{231}{4} = 57.75 \frac{ft}{sec^2}$ ◄ $= 1.79\,g$

Example #2. A golfer tees off on a par 3 hole, where the tee is 100 ft above the green. When the golf ball is hit, it has an initial upward velocity of $75 \frac{ft}{sec}$.

(a) What is the maximum height (above the green) of the ball?

(b) How long after the ball is hit does the ball land on the green?

Use $g = 32 \frac{ft}{sec^2}$.

SOLUTION:

(a) $x = x_0 + v_0 t - \frac{1}{2} g t^2 = 100 + 75t - 16t^2 = -16t^2 + 75t + 100$

$t_{max} = \dfrac{-b}{2a} = -\dfrac{75}{2(-16)} = 2.34375\,sec, \quad x_{max} = x(2.34375) = 187.89\,ft$ ◄

(b) $0 = -16t^2 + 75t + 100, \quad t = \dfrac{-75 \pm \sqrt{75^2 - 4(-16)(100)}}{2(-16)} = \dfrac{-75 \pm \sqrt{12{,}025}}{-32}$

$t = -1.08\,sec$ ✗ $\quad t = 5.77\,sec$ ◄

3.4. Equations of Motion

CLASS WORK

1) Repeat Example #1 where the position of the car is given by

$$x = \frac{55}{27}\left(24t^2 - t^3\right),$$

again, where x is in feet and t is in seconds. Note that the 0.25 mile drag race is completed in 6 seconds.

2) Mr. Lody hits a home run. When the baseball is hit, it has an initial upward velocity of $60\frac{ft}{sec}$. The ball lands in the stands which are 50 ft above home plate. Use $g = 32\frac{ft}{sec^2}$.

(a) Find the maximum height of the ball.

(b) How long after the ball is hit does it land in the stands?

SOLUTIONS

1)

a) $v = \frac{dx}{dt} = \frac{55}{27}\left(48t - 3t^2\right) = -\frac{55}{9}t^2 + \frac{880}{9}t$ ◄

b) $a = \frac{dv}{dt} = -\frac{110}{9}t + \frac{880}{9}$ ◄

c) $t_{max} = \frac{-b}{2a} = \frac{-\frac{880}{9}}{2\left(-\frac{55}{9}\right)} = \frac{880}{9} \cdot \frac{9}{110} = 8\ sec \Rightarrow$

$v_{max} = v(6) = 366\frac{2}{3}\ \frac{ft}{sec}$ ◄ $\quad \frac{366\frac{2}{3}\cancel{ft}}{sec}\left(\frac{1mi}{5280\cancel{ft}}\right)\left(\frac{3600 sec}{1hr}\right) = 250\frac{mi}{hr}$ ◄

d) $v_{ave} = \frac{x(6) - x(0)}{6-0} = \frac{1320}{6} = 220\frac{ft}{sec}$ ◄

$\frac{220\cancel{ft}}{sec}\left(\frac{1mi}{5280\cancel{ft}}\right)\left(\frac{3600 sec}{1hr}\right) = 150\frac{mi}{hr}$ ◄

e) $a_{max} = a(0) = \frac{880}{9} = 97\frac{7}{9}\ \frac{ft}{sec^2}$ ◄ $= 3.04g$

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2) $x = 60t - 16t^2$

a) $t_{max} = -\frac{b}{2a} = -\frac{60}{2(-16)} = 1.875\ sec$, $x(1.875) = 56.25\ ft$

b) $50 = 60t - 16t^2$, $-16t^2 + 60t - 50 = 0$

$$t = \frac{-60 \pm \sqrt{60^2 - 4(-16)(-50)}}{2(-16)} = \frac{-60 \pm \sqrt{400}}{-32} = \frac{-60 \pm 20}{-32}$$

$t = 1.25\ sec$ ✗ $t = 2.5\ sec$

on way up