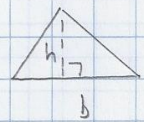
3.5. Two Important Limits

Recall

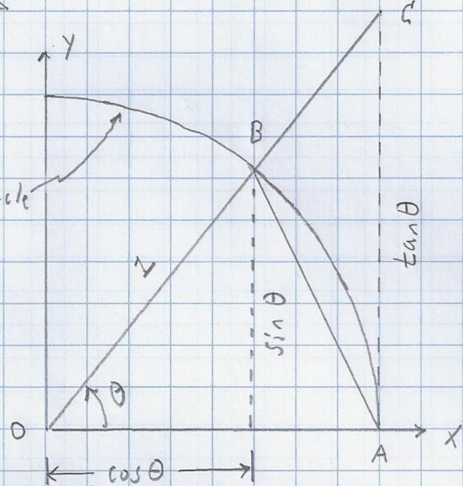
Area of sector $= \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(r\theta)r = \frac{1}{2}r^2\theta$

θ is in radians



Area triangle $= \frac{1}{2}bh$

Consider



Area triangle $OAB = \frac{1}{2}(1)\sin\theta = \frac{1}{2}\sin\theta$

Area sector $OAB = \frac{1}{2}(1)^2\theta = \frac{1}{2}\theta$

Area triangle $OAC = \frac{1}{2}(1)\tan\theta = \frac{1}{2}\tan\theta$

Area triangle OAB < Area sector OAB < Area triangle OAC

$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$, $\sin\theta < \theta < \tan\theta$, $1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$,

$1 > \frac{\sin\theta}{\theta} > \cos\theta$. Take the limit as $\theta \to 0$ $\Rightarrow$

$1 > \lim_{\theta \to 0} \frac{\sin\theta}{\theta} > \lim_{\theta \to 0} \cos\theta$, $1 > \lim_{\theta \to 0} \frac{\sin\theta}{\theta} > 1$.

$$\Rightarrow \boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1}$$

3.5. Two Important Limits

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = \lim_{\theta \to 0} \frac{(\cos\theta - 1)}{\theta} \cdot \frac{(\cos\theta + 1)}{(\cos\theta + 1)} = \lim_{\theta \to 0} \frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)}$$

$$\cos^2\theta = 1 - \sin^2\theta \Rightarrow \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = \lim_{\theta \to 0} \frac{-\sin^2\theta}{\theta(\cos\theta + 1)} =$$

$$= -\lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{(\cos\theta + 1)} = -1 \cdot \frac{0}{1+1} = 0$$

$$\boxed{\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0}$$