# 3.5. Derivatives of the Trigonometric Functions

Recall the angle sum formula for sine...

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \text{, or}$$
$$\sin(x+h) = \sin x\cos h + \cos x\sin h$$

$$\frac{d\sin x}{dx} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h} =$$

$$= \lim_{h\to 0}\frac{\sin x(\cos h - 1) + \cos x\sin h}{h} = \lim_{h\to 0}\left[\frac{\sin x(\cos h-1)}{h} + \frac{\cos x\sin h}{h}\right] =$$

$$= \sin x\lim_{h\to 0}\frac{\cos h-1}{h} + \cos x\lim_{h\to 0}\frac{\sin h}{h} = \sin x\cdot 0 + \cos x\cdot 1$$

$$\boxed{\frac{d\sin x}{dx} = \cos x}$$

Recall the angle sum formula for cosine...

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \text{, or}$$
$$\cos(x+h) = \cos x\cos h - \sin x\sin h$$

$$\frac{d\cos x}{dx} = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h} =$$

$$= \lim_{h\to 0}\frac{\cos x(\cos h-1) - \sin x\sin h}{h} = \lim_{h\to 0}\left[\frac{\cos x(\cos h-1)}{h} - \frac{\sin x\sin h}{h}\right] =$$

$$= \cos x\lim_{h\to 0}\frac{\cos h-1}{h} - \sin x\lim_{h\to 0}\frac{\sin h}{h} = \cos x\cdot 0 - \sin x\cdot 1$$

$$\boxed{\frac{d\cos x}{dx} = -\sin x}$$

# 3.5. Derivatives of the Trigonometric Functions

**Example.** Use the Quotient Rule to calculate

a) $\frac{d\tan x}{dx}$  b) $\frac{d\sec x}{dx}$

**SOLUTION:**

a) $$\frac{d\tan x}{dx} = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2} = \frac{\cos x \cdot \cos x - \sin x \cdot -\sin x}{\cos^2 x} =$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x \blacktriangleleft$$

b) $$\frac{d\sec x}{dx} = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2} = \frac{0 \cdot \cos x - 1 \cdot -\sin x}{\cos^2 x} =$$

$$= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x \blacktriangleleft$$

## CLASS WORK

Use the Quotient Rule to calculate

1) $\frac{d\cot x}{dx}$  2) $\frac{d\csc x}{dx}$

**SOLUTIONS:**

1) $$\frac{d\cot x}{dx} = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2} = \frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x} =$$

$$= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x \blacktriangleleft$$

2) $$\frac{d\csc x}{dx} = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2} = \frac{0 \cdot \sin x - 1 \cdot \cos x}{\sin^2 x} =$$

$$= -\frac{\cos x}{\sin^2 x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x \blacktriangleleft$$