Section 4.1 Exercises

In Exercises 1–8, use the given substitution and the Chain Rule to find dy/dx.

1. $y = \sin(3x+1),\ u = 3x+1$

2. $y = \sin(7-5x),\ u = 7-5x$

3. $y = \cos(\sqrt{3}x),\ u = \sqrt{3}x$

4. $y = \tan(2x - x^3),\ u = 2x - x^3$

5. $y = \left(\dfrac{\sin x}{1+\cos x}\right)^2, \quad u = \dfrac{\sin x}{1+\cos x}$

6. $y = 5\cot\left(\dfrac{2}{x}\right),\ u = \dfrac{2}{x}$

7. $y = \cos(\sin x),\ u = \sin x$

8. $y = \sec(\tan x),\ u = \tan x$

In Exercises 9–12, an object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = s(t)$. Find the velocity of the object as a function of t.

9. $s = \cos\left(\dfrac{\pi}{2} - 3t\right)$

10. $s = t\cos(\pi - 4t)$

11. $s = \dfrac{4}{3\pi}\sin 3t + \dfrac{4}{5\pi}\cos 5t$

12. $s = \sin\left(\dfrac{3\pi}{2}t\right) + \cos\left(\dfrac{7\pi}{4}t\right)$

In Exercises 13–24, find dy/dx.

13. $y = (x + \sqrt{x})^{-2}$

14. $y = (\csc x + \cot x)^{-1}$

15. $y = \sin^{-5} x - \cos^3 x$

16. $y = x^3(2x-5)^4$

17. $y = \sin^3 x \tan 4x$

18. $y = 4\sqrt{\sec x + \tan x}$

19. $y = \dfrac{3}{\sqrt{2x+1}}$

20. $y = \dfrac{x}{\sqrt{1+x^2}}$

21. $y = \sin^2(3x-2)$

22. $y = (1+\cos 2x)^2$

23. $y = (1+\cos^2 7x)^3$

24. $y = \sqrt{\tan 5x}$

In Exercises 25–28, find $dr/d\theta$.

25. $r = \tan(2-\theta)$

26. $r = \sec 2\theta \tan 2\theta$

27. $r = \sqrt{\theta \sin \theta}$

28. $r = 2\theta\sqrt{\sec \theta}$

In Exercises 29–32, find y''.

29. $y = \tan x$

30. $y = \cot x$

31. $y = \cot(3x-1)$

32. $y = 9\tan(x/3)$

In Exercises 33–38, find the value of $(f \circ g)'$ at the given value of x.

33. $f(u) = u^5 + 1, \quad u = g(x) = \sqrt{x}, \quad x = 1$

34. $f(u) = 1 - \dfrac{1}{u}, \quad u = g(x) = \dfrac{1}{1-x}, \quad x = -1$

35. $f(u) = \cot\dfrac{\pi u}{10}, \quad u = g(x) = 5\sqrt{x}, \quad x = 1$

36. $f(u) = u + \dfrac{1}{\cos^2 u}, \quad u = g(x) = \pi x, \quad x = \dfrac{1}{4}$

37. $f(u) = \dfrac{2u}{u^2+1}, \quad u = g(x) = 10x^2 + x + 1, \quad x = 0$

38. $f(u) = \left(\dfrac{u-1}{u+1}\right)^2, \quad u = g(x) = \dfrac{1}{x^2} - 1, \quad x = -1$

What happens if you can write a function as a composite in different ways? Do you get the same derivative each time? The Chain Rule says you should. Try it with the functions in Exercises 39 and 40.

39. Find dy/dx if $y = \cos(6x + 2)$ by writing y as a composite with

(a) $y = \cos u$ and $u = 6x + 2$.

(b) $y = \cos 2u$ and $u = 3x + 1$.

40. Find dy/dx if $y = \sin(x^2 + 1)$ by writing y as a composite with

(a) $y = \sin(u + 1)$ and $u = x^2$.

(b) $y = \sin u$ and $u = x^2 + 1$.

In Exercises 41–48, find the equation of the line tangent to the curve at the point defined by the given value of t.

41. $x = 2\cos t, \quad y = 2\sin t, \quad t = \pi/4$

42. $x = \sin 2\pi t, \quad y = \cos 2\pi t, \quad t = -1/6$

43. $x = \sec^2 t - 1, \quad y = \tan t, \quad t = -\pi/4$

44. $x = \sec t, \quad y = \tan t, \quad t = \pi/6$

45. $x = t, \quad y = \sqrt{t}, \quad t = 1/4$

46. $x = 2t^2 + 3, \quad y = t^4, \quad t = -1$

47. $x = t - \sin t, \quad y = 1 - \cos t, \quad t = \pi/3$

48. $x = \cos t, \quad y = 1 + \sin t, \quad t = \pi/2$

49. Let $x = t^2 + t$, and let $y = \sin t$.

(a) Find dy/dx as a function of t.

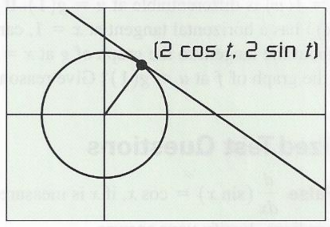
(b) Find $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)$ as a function of t.

(c) Find $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ as a function of t.

Use the Chain Rule and your answer from part (b).

(d) Which of the expressions in parts (b) and (c) is d^2y/dx^2?

50. A circle of radius 2 and center $(0, 0)$ can be parametrized by the equations $x = 2\cos t$ and $y = 2\sin t$. Show that for any value of t, the line tangent to the circle at $(2\cos t, 2\sin t)$ is perpendicular to the radius.



51. Let $s = \cos\theta$. Evaluate ds/dt when $\theta = 3\pi/2$ and $d\theta/dt = 5$.

52. Let $y = x^2 + 7x - 5$. Evaluate dy/dt when $x = 1$ and $dx/dt = 1/3$.

53. What is the largest value possible for the slope of the curve $y = \sin(x/2)$?

54. Write an equation for the tangent to the curve $y = \sin mx$ at the origin.

55. Find the lines that are tangent and normal to the curve $y = 2\tan(\pi x/4)$ at $x = 1$. Support your answer graphically.

56. *Working with Numerical Values* Suppose that functions f and g and their derivatives have the following values at $x = 2$ and $x = 3$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	$1/3$	-3
3	3	-4	2π	5

Evaluate the derivatives with respect to x of the following combinations at the given value of x.

(a) $2f(x)$ at $x = 2$ (b) $f(x) + g(x)$ at $x = 3$

(c) $f(x) \cdot g(x)$ at $x = 3$ (d) $f(x)/g(x)$ at $x = 2$

(e) $f(g(x))$ at $x = 2$ (f) $\sqrt{f(x)}$ at $x = 2$

(g) $1/g^2(x)$ at $x = 3$ (h) $\sqrt{f^2(x) + g^2(x)}$ at $x = 2$

57. *Extension of Example 8* Show that $\dfrac{d}{dx}\cos(x°)$ is $-\dfrac{\pi}{180}\sin(x°)$.

58. *Working with Numerical Values* Suppose that the functions f and g and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	$1/3$
1	3	-4	$-1/3$	$-8/3$

Evaluate the derivatives with respect to x of the following combinations at the given value of x.

(a) $5f(x) - g(x), \quad x = 1$ (b) $f(x)g^3(x), \quad x = 0$

(c) $\dfrac{f(x)}{g(x) + 1}, \quad x = 1$ (d) $f(g(x)), \quad x = 0$

(e) $g(f(x)), \quad x = 0$ (f) $(g(x) + f(x))^{-2}, \quad x = 1$

(g) $f(x + g(x)), \quad x = 0$

59. *Orthogonal Curves* Two curves are said to cross at right angles if their tangents are perpendicular at the crossing point. The technical word for "crossing at right angles" is **orthogonal.** Show that the curves $y = \sin 2x$ and $y = -\sin(x/2)$ are orthogonal at the origin. Draw both graphs and both tangents in a square viewing window.

60. Writing to Learn Explain why the Chain Rule formula

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

is not simply the well-known rule for multiplying fractions.

61. ***Running Machinery Too Fast*** Suppose that a piston is moving straight up and down and that its position at time t seconds is

$$s = A\cos(2\pi bt),$$

with A and b positive. The value of A is the amplitude of the motion, and b is the frequency (number of times the piston moves up and down each second). What effect does doubling the frequency have on the piston's velocity, acceleration, and jerk? (Once you find out, you will know why machinery breaks when you run it too fast.)

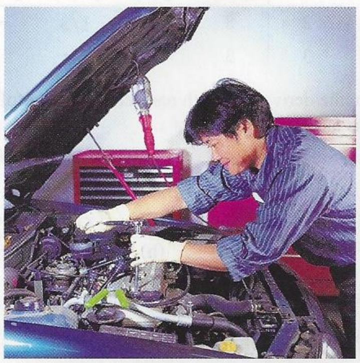


Figure 4.5 The internal forces in the engine get so large that they tear the engine apart when the velocity is too great.

62. Group Activity ***Tempe Temperatures.*** The graph in Figure 4.6 shows the variation in average daily temperature in Tempe, Arizona, during a typical 365-day year. The equation that approximates the Fahrenheit temperature on day x is

$$y = 19.3\sin\left[\frac{2\pi}{365}(x - 101)\right] + 70.$$

(a) On approximately what day of the year does the daily temperature show the greatest increase from the previous day?

(b) About how many degrees per day is the temperature increasing at that time of the year?

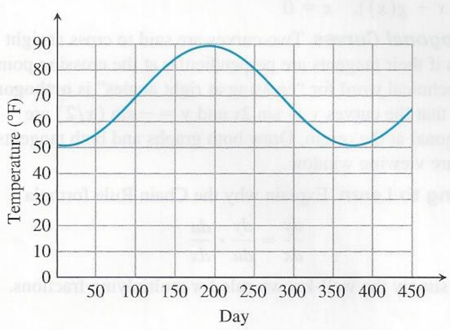


Figure 4.6 Average daily temperatures in Tempe, AZ, for a typical 365-day year are modeled by a sinusoid (Exercise 62).

63. ***Particle Motion*** The position of a particle moving along a coordinate line is $s = \sqrt{1 + 4t}$, with s in meters and t in seconds. Find the particle's velocity and acceleration at $t = 6$ sec.

64. ***Constant Acceleration*** Suppose the velocity of a falling body is $v = k\sqrt{s}$ m/sec (k a constant) at the instant the body has fallen s meters from its starting point. Show that the body's acceleration is constant.

65. ***Falling Meteorite*** The velocity of a heavy meteorite entering the earth's atmosphere is inversely proportional to $\sqrt{s}$ when it is s kilometers from the earth's center. Show that the meteorite's acceleration is inversely proportional to s^2.

66. ***Particle Acceleration*** A particle moves along the x-axis with velocity $dx/dt = f(x)$. Show that the particle's acceleration is $f(x)f'(x)$.

67. ***Temperature and the Period of a Pendulum*** For oscillations of small amplitude (short swings), we may safely model the relationship between the period T and the length L of a simple pendulum with the equation

$$T = 2\pi\sqrt{\frac{L}{g}},$$

where g is the constant acceleration of gravity at the pendulum's location. If we measure g in centimeters per second squared, we measure L in centimeters and T in seconds. If the pendulum is made of metal, its length will vary with temperature, either increasing or decreasing at a rate that is roughly proportional to L. In symbols, with u being temperature and k the proportionality constant,

$$\frac{dL}{du} = kL.$$

Assuming this to be the case, show that the rate at which the period changes with respect to temperature is $kT/2$.

68. Writing to Learn ***Chain Rule*** Suppose that $f(x) = x^2$ and $g(x) = |x|$. Then the composites

$$(f \circ g)(x) = |x|^2 = x^2 \quad \text{and} \quad (g \circ f)(x) = |x^2| = x^2$$

are both differentiable at $x = 0$ even though g itself is not differentiable at $x = 0$. Does this contradict the Chain Rule? Explain.

69. ***Tangents*** Suppose that $u = g(x)$ is differentiable at $x = 1$ and that $y = f(u)$ is differentiable at $u = g(1)$. If the graph of $y = f(g(x))$ has a horizontal tangent at $x = 1$, can we conclude anything about the tangent to the graph of g at $x = 1$ or the tangent to the graph of f at $u = g(1)$? Give reasons for your answer.

Standardized Test Questions

70. True or False $\frac{d}{dx}(\sin x) = \cos x$, if x is measured in degrees or radians. Justify your answer.

71. True or False The slope of the normal line to the curve $x = 3\cos t$, $y = 3\sin t$ at $t = \pi/4$ is -1. Justify your answer.

72. Multiple Choice Which of the following is dy/dx if $y = \tan(4x)$?

(A) $4\sec(4x)\tan(4x)$ **(B)** $\sec(4x)\tan(4x)$ **(C)** $4\cot(4x)$
(D) $\sec^2(4x)$ **(E)** $4\sec^2(4x)$

73. Multiple Choice Which of the following is dy/dx if $y = \cos^2(x^3 + x^2)$?

(A) $-2(3x^2 + 2x)$

(B) $-(3x^2 + 2x)\cos(x^3 + x^2)\sin(x^3 + x^2)$

(C) $-2(3x^2 + 2x)\cos(x^3 + x^2)\sin(x^3 + x^2)$

(D) $2(3x^2 + 2x)\cos(x^3 + x^2)\sin(x^3 + x^2)$

(E) $2(3x^2 + 2x)$

In Exercises 74 and 75, use the curve defined by the parametric equations $x = t - \cos t$, $y = -1 + \sin t$.

74. Multiple Choice Which of the following is an equation of the tangent line to the curve at $t = 0$?

(A) $y = x$ (B) $y = -x$ (C) $y = x + 2$

(D) $y = x - 2$ (E) $y = -x - 2$

75. Multiple Choice At which of the following values of t is $dy/dx = 0$?

(A) $t = \pi/4$ (B) $t = \pi/2$ (C) $t = 3\pi/4$

(D) $t = \pi$ (E) $t = 2\pi$

Explorations

76. *The Derivative of sin 2x* Graph the function $y = 2\cos 2x$ for $-2 \le x \le 3.5$. Then, on the same screen, graph

$$y = \frac{\sin 2(x + h) - \sin 2x}{h}$$

for $h = 1.0, 0.5$, and 0.2. Experiment with other values of h, including negative values. What do you see happening as $h \to 0$? Explain this behavior.

77. *The Derivative of* cos (x^2) Graph $y = -2x\sin(x^2)$ for $-2 \le x \le 3$. Then, on the same screen, graph

$$y = \frac{\cos[(x + h)^2] - \cos(x)^2}{h}$$

for $h = 1.0, 0.7$, and 0.3. Experiment with other values of h. What do you see happening as $h \to 0$? Explain this behavior.

Extending the Ideas

78. *Absolute Value Functions* Let u be a differentiable function of x.

(a) Show that $\dfrac{d}{dx}|u| = u'\dfrac{u}{|u|}$.

(b) Use part (a) to find the derivatives of $f(x) = |x^2 - 9|$ and $g(x) = |x|\sin x$.

79. *Geometric and Arithmetic Mean* The geometric mean of u and v is $G = \sqrt{uv}$ and the arithmetic mean is $A = (u + v)/2$. Show that if $u = x$, $v = x + c$, c a real number, then

$$\frac{dG}{dx} = \frac{A}{G}.$$