4.1. Chain Rule

2-Link Chain

$$f(x) = f(g(x)) \quad \Rightarrow \quad \boxed{\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx} = \frac{df}{\cancel{dg}}\frac{\cancel{dg}}{dx}}$$

Example #1. For $f(x) = 3x^2 + 2x + 1$ and $g(x) = 4x + 7$, calculate $\frac{df}{dx}$ for $f = f(g(x))$ two ways, i.e., by

(a) composing then differentiating, and

(b) using the chain rule and then composing.

SOLUTION:

(a) $f(g) = 3g^2 + 2g + 1 = 3(4x+7)^2 + 2(4x+7) + 1 =$
$= 3(16x^2 + 56x + 49) + 2(4x+7) + 1 =$
$= 48x^2 + 168x + 147 + 8x + 14 + 1 = 48x^2 + 176x + 162$

$$\frac{df}{dx} = 96x + 176$$

(b) $f(g) = 3g^2 + 2g + 1$, $\frac{df}{dg} = 6g + 2$; $\frac{dg}{dx} = 4$,

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx} = (6g+2)(4) = 24g + 8 = 24(4x+7) + 8 = 96x + 168 + 8 =$$
$$= 96x + 176$$

Example #2. Find $f'(x)$ for $f(x) = \sin(3x^2 + 8)$.

SOLUTION:

Way #1. $f(g) = \sin g$, $g(x) = 3x^2 + 8$. $\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx} = \cos g \cdot 6x =$
$= 6x\cos(3x^2 + 8)$

Way #2. (outside-in) $f(x) = \sin(3x^2+8) = \cos(3x^2+8) \cdot 6x =$

derivative of outside function

derivative of inside function

$= 6x\cos(3x^2+8)$

3-Link Chain

$f(x) = f(g(h(x))) \Rightarrow \boxed{\dfrac{df}{dx} = \dfrac{df}{dg}\dfrac{dg}{dh}\dfrac{dh}{dx} = \dfrac{df}{\cancel{dg}}\dfrac{\cancel{dg}}{dh}\dfrac{dh}{dx}}$

Example #3. Find $f'(x)$ for $f(x) = \sin^2(8x^2+7)$.

Solution:

Way #1. $f(g) = g^2$, $g(h) = \sin h$, $h(x) = 8x^2+7$

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dh}\frac{dh}{dx} = 2g \cdot \cosh \cdot 16x = 2\sin h \cdot \cos h \cdot 16x =$$

$$= 32x\sin(8x^2+7)\cos(8x^2+7)$$

Way #2. (outside-in)

$$\frac{df}{dx} = \underbrace{2\sin(8x^2+7)}_{\text{derivative of outside function}} \cdot \underbrace{\cos(8x^2+7)}_{\text{derivative of middle function}} \cdot \underbrace{16x}_{\text{derivative of inside function}} = 32x\sin(8x^2+7)\cos(8x^2+7)$$

4.1. Chain Rule

CLASS WORK

1) For $f(x) = 3x^2 - 7x + 5$ and $g(x) = 5x - 3$, calculate $\frac{df}{dx}$ for $f = f(g(x))$, two ways, i.e.,

a) composing then differentiating, and
b) using the chain rule then composing.

2) Use the "outside-in" method to calculate $f'(x)$ for

a) $f(x) = \tan(5x^3 + 4x)$
b) $f(x) = \sec^2(3x^5 + 7x)$

SOLUTIONS

1)
a)

$$f(g) = 3g^2 - 7g + 5 = 3(5x-3)^2 - 7(5x-3) + 5 =$$
$$= 3(25x^2 - 30x + 9) - 7(5x-3) + 5 =$$
$$= 75x^2 - 90x + 27 - 35x + 21 + 5 = 75x^2 - 125x + 53$$
$$\frac{df}{dx} = 150x - 125$$

b)

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx} = (6g - 7)(5) = 30g - 35 = 30(5x-3) - 35 =$$
$$= 150x - 90 - 35 = 150x - 125$$

2)
a)

$$\frac{df}{dx} = \sec^2(5x^3 + 4x) \cdot (15x^2 + 4) = (15x^2 + 4)\sec^2(5x^3 + 4x)$$

b)

$$\frac{df}{dx} = 2\sec(3x^5 + 7x) \cdot \sec(3x^5 + 7x)\tan(3x^5 + 7x) \cdot (15x^4 + 7) =$$
$$= 2(15x^4 + 7)\sec^2(3x^5 + 7x)\tan(3x^5 + 7x)$$