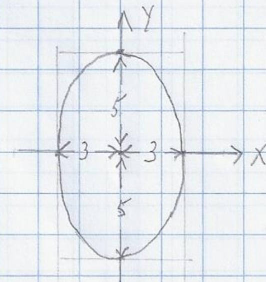
4.2. Implicit Differentiation

- **Implicitly Defined Function** ≡ not solved for y, e.g., $y + \tan(xy) = 0$. An implicitly defined function usually cannot be solved for y.

- **Implicit Differentiation** ≡ differentiate both sides of an implicitly defined function with respect to x and use the Chain Rule.

Example #1. Differentiate the ellipse $\frac{x^2}{3^2} + \frac{y^2}{5^2} = 1$



SOLUTION:

$$\frac{x^2}{9} + \frac{y^2}{25} = 1 \qquad 25x^2 + 9y^2 = 25 \cdot 9 = 225$$

$$25\frac{dx^2}{dx} + 9\frac{dy^2}{dx} = 0 \quad , \quad 25 \cdot 2x + 9 \cdot \frac{dy^2}{dy}\frac{dy}{dx} = 0,$$

$$25 \cdot \cancel{2}x + 9 \cdot 2y\frac{dy}{dx} = 0 \quad , \quad 9y\frac{dy}{dx} = -25x, \quad \frac{dy}{dx} = -\frac{25x}{9y}$$

Example #2. Differentiate $x^3 + xy^2 + xy^3 = 8$.

SOLUTION:

$$\frac{dx^3}{dx} + \frac{d}{dx}(xy^2) + \frac{d}{dx}(xy^3) = 0$$

$$3x^2 + 1 \cdot y^2 + x\frac{dy^2}{dx} + 1 \cdot y^3 + x\frac{dy^3}{dx} = 0$$

$$3x^2 + y^2 + x\frac{dy^2}{dy}\frac{dy}{dx} + y^3 + x\frac{dy^3}{dy}\frac{dy}{dx} = 0$$

$$3x^2 + y^2 + x \cdot 2y\frac{dy}{dx} + y^3 + x \cdot 3y^2\frac{dy}{dx} = 0$$

$$(3x^2 + y^2 + y^3) + \frac{dy}{dx}(2xy + 3xy^2) = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{3x^2 + y^2 + y^3}{2xy + 3xy^2}$$

Example #3. Differentiate $x + y\sec(xy) = 0$.

SOLUTION:

$$\frac{dx}{dx} + \frac{d}{dx}\left[y\sec(xy)\right] = 0$$

$$1 + \frac{dy}{dx}\sec(xy) + y\frac{d}{dx}\sec(xy) = 0$$

4.2: Implicit Differentiation

$$1+\frac{dy}{dx}\sec(xy)+y\sec(xy)\tan(xy)\frac{d}{dx}(xy)=0$$

$$1+\frac{dy}{dx}\sec(xy)+y\sec(xy)\tan(xy)\left[1\cdot y+x\frac{dy}{dx}\right]=0$$

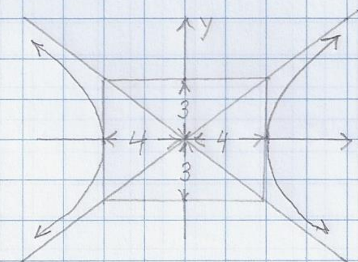
$$1+\frac{dy}{dx}\sec(xy)+y^2\sec(xy)\tan(xy)+xy\sec(xy)\tan(xy)\frac{dy}{dx}=0$$

$$1+y^2\sec(xy)\tan(xy)+\frac{dy}{dx}\sec(xy)\left[1+xy\tan(xy)\right]=0$$

$$\frac{dy}{dx}=-\frac{1+y^2\sec(xy)\tan(xy)}{\sec(xy)\left[1+xy\tan(xy)\right]}$$

CLASS WORK

1) Differentiate the hyperbola $\frac{x^2}{4^2}-\frac{y^2}{3^2}=1$.



2) Differentiate

a) $x^4y^2+x^3y^5=10$ b) $y+x\cos(xy)=0$

SOLUTIONS:

1) $\frac{x^2}{16}-\frac{y^2}{9}=1$, $9x^2-16y^2=144$, $9\frac{dx^2}{dx}-16\frac{dy^2}{dx}=0$

$9\cdot 2x-16\frac{dy^2}{dy}\frac{dy}{dx}=0$, $9\cdot\cancel{2}x-16\cdot\cancel{2}y\frac{dy}{dx}=0$

$9x=16y\frac{dy}{dx}$, $\frac{dy}{dx}=\frac{9x}{16y}$

2) a) $\frac{d}{dx}(x^4y^2)+\frac{d}{dx}(x^3y^5)=0$, $\frac{dx^4}{dx}y^2+x^4\frac{dy^2}{dx}+\frac{dx^3}{dx}y^5+x^3\frac{dy^5}{dx}=0$,

$4x^3y^2+x^4\frac{dy^2}{dy}\frac{dy}{dx}+3x^2y^5+x^3\frac{dy^5}{dy}\frac{dy}{dx}=0$,

$$4x^3y^2 + x^4 \cdot 2y\frac{dy}{dx} + 3x^2y^5 + x^3 \cdot 5y^4\frac{dy}{dx} = 0$$

$$4x^3y^2 + 3x^2y^5 + \frac{dy}{dx}\left(2x^4y + 5x^3y^4\right) = 0$$

$$\frac{dy}{dx} = -\frac{4x^3y^2 + 3x^2y^5}{2x^4y + 5x^3y^4}$$

2) b)

$$\frac{dy}{dx} + \frac{d}{dx}\left[x\cos(xy)\right] = 0, \quad \frac{dy}{dx} + 1 \cdot \cos(xy) + x \cdot -\sin(xy)\frac{d}{dx}(xy) = 0$$

$$\frac{dy}{dx} + \cos(xy) - x\sin(xy)\left[1 \cdot y + x\frac{dy}{dx}\right] = 0,$$

$$\frac{dy}{dx} + \cos(xy) - xy\sin(xy) - x^2\sin(xy)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left[1 - x^2\sin(xy)\right] = xy\sin(xy) - \cos(xy)$$

$$\frac{dy}{dx} = \frac{xy\sin(xy) - \cos(xy)}{1 - x^2\sin(xy)}$$