4.2. Implicit Differetiation – II

Example:

(a) Use the Quadratic Formula to solve the equation of a parabola

$$16x^2+9y^2+24xy+30x-40y-229=0$$

for y. Graph the parabola.

(b) Implicitly differentiate the equation of the parabola.

(c) Find the equation of the line tangent to the parabola at $(x,y)=(2,3)$. Include the tangent line on the graph from part (a).

(d) Find the equation of the line normal to the parabola at $(x,y)=(2,3)$. Include the normal line on the graph from part (a).

(e) Find the domain of the parabola.

(f) Find the range of the parabola.

SOLUTION:

(a) $9y^2+24xy-40y+16x^2+30x-229=0$

$$9y^2+8(3x-5)y+(16x^2+30x-229)=0$$

$$b^2-4ac = 64(3x-5)^2-4(9)(16x^2+30x-229)$$

$$= 64(9x^2-30x+25)-36(16x^2+30x-229)$$

$$= \cancel{576x^2}-1920x+1600-\cancel{576x^2}-1080x+8244$$

$$= 9844-3000x = 4(2461-750x)$$

$$\sqrt{b^2-4ac} = 2\sqrt{2461-750x}$$

$$y=\frac{-8(3x-5)\pm 2\sqrt{2461-750x}}{2(9)} = -\frac{4}{9}(3x-5)\pm\frac{1}{9}\sqrt{2461-750x}$$

4.2. Implicit Differentiation – II

(b) $16x^2 + 9y^2 + 24xy + 30x - 40y - 229 = 0$

$16 \cdot 2x + 9\frac{dy^2}{dx} + 24\frac{d}{dx}[xy] + 30 - 40\frac{dy}{dx} = 0$

$32x + 9 \cdot 2y\frac{dy}{dx} + 24\left[1 \cdot y + x\frac{dy}{dx}\right] + 30 - 40\frac{dy}{dx} = 0$

$32x + 18y\frac{dy}{dx} + 24y + 24x\frac{dy}{dx} + 30 - 40\frac{dy}{dx} = 0$

$16x + 9y\frac{dy}{dx} + 12y + 12x\frac{dy}{dx} + 15 - 20\frac{dy}{dx} = 0$

$16x + 12y + 15 + \frac{dy}{dx}[9y + 12x - 20] = 0$

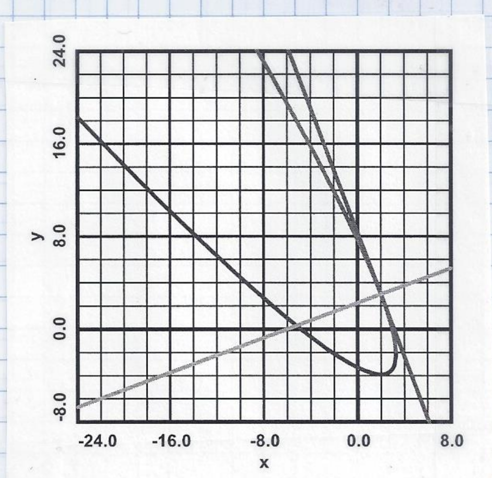
$\frac{dy}{dx} = -\frac{16x + 12y + 15}{12x + 9y - 20}$

(c) $\left.\frac{dy}{dx}\right|_{(2,3)} = -\frac{16(2) + 12(3) + 15}{12(2) + 9(3) - 20} = -\frac{83}{31}$, $y = -\frac{83}{31}x + b$, $3 = -\frac{83}{31}(2) + b$,

$b = \frac{259}{31}$, $y = -\frac{83}{31}x + \frac{259}{31}$

(d) The slope of the normal is the opposite reciprocal $\Rightarrow y = \frac{31}{83}x + b$,

$3 = \frac{31}{83}(2) + b$, $b = \frac{187}{83}$, $y = \frac{31}{83}x + \frac{187}{83}$



(e) Where the slope of the parabola is infinite gives the x-value for finding the domain. So, the denominator of the slope is zero.

$12x + 9y - 20 = 0$, $9y = -12x + 20$, $y = -\frac{4}{3}x + \frac{20}{9}$, $y^2 = \frac{16}{9}x^2 - \frac{160}{27}x + \frac{400}{81}$

Put these into the original equation $16x^2 + 9y^2 + 24xy + 30x - 40y - 229 = 0 \Rightarrow$

$16x^2 + 9\left(\frac{16}{9}x^2 - \frac{160}{27}x + \frac{400}{81}\right) + 24x\left(-\frac{4}{3}x + \frac{20}{9}\right) + 30x - 40\left(-\frac{4}{3}x + \frac{20}{9}\right) - 229 = 0$

$\cancel{16x^2} + \cancel{16x^2} - \cancel{\frac{160}{3}x} + \frac{400}{9} - \cancel{32x^2} + \cancel{\frac{160}{3}x} + 30x + \frac{160}{3}x - \frac{800}{9} - 229 = 0$

$\frac{250}{3}x - \frac{2461}{9} = 0$, $750x - 2461 = 0$, $x = \frac{2461}{750} \approx 3.281$

$x \in (-\infty, 3.281]$ ◄

(f) Where the slope of the parabola is zero gives the y-value for finding the range. So, the numerator of the slope is zero.

$16x + 12y + 15 = 0$, $16x = -12y - 15$, $x = -\frac{3}{4}y - \frac{15}{16}$, $x^2 = \frac{9}{16}y^2 + \frac{45}{32}y + \frac{225}{256}$

Put these into the original equation $\Rightarrow$

$16\left(\frac{9}{16}y^2 + \frac{45}{32}y + \frac{225}{256}\right) + 9y^2 + 24\left(-\frac{3}{4}y - \frac{15}{16}\right)y + 30\left(-\frac{3}{4}y - \frac{15}{16}\right) - 40y - 229 = 0$

$\cancel{9y^2} + \cancel{\frac{45}{2}y} + \frac{225}{16} + \cancel{9y^2} - \cancel{18y^2} - \cancel{\frac{45}{2}y} - \frac{45}{2}y - \frac{225}{8} - 40y - 229 = 0$

$-\frac{125}{2}y - \frac{3889}{16} = 0$, $1000y + 3889 = 0$, $y = -\frac{3889}{1000} = -3.889$

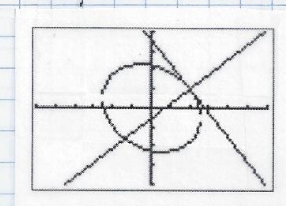
$y \in [-3.889, \infty)$ ◄

CLASS WORK

Repeat the above example, except use the equation of an ellipse

$4x^2 + 5y^2 + 2xy - 25 = 0$.

Find the tangent and normal lines at the point $(x, y) = (2, 1)$.
Graph the ellipse and tangent and normal lines with your calculator using the window $x \in [-6, 6]$ and $y \in [-4, 4]$. Show me the graph.



SOLUTION

(a) $5y^2 + (2x)y + (4x^2-25) = 0$

$b^2-4ac = 4x^2 - 4(5)(4x^2-25) = 4x^2 - 20(4x^2-25) = 500 - 76x^2 = 4(125-19x^2)$

$\sqrt{b^2-4ac} = 2\sqrt{125-19x^2}$

$y = \dfrac{-2x \pm 2\sqrt{125-19x^2}}{2(5)} = -\frac{1}{5}x \pm \frac{1}{5}\sqrt{125-19x^2}$

(b) $4\cdot 2x + 5\dfrac{dy^2}{dx} + 2\dfrac{d}{dx}(xy) = 0$, $4\cdot 2x + 5\dfrac{dy^2}{dy}\dfrac{dy}{dx} + 2\left(1\cdot y + x\dfrac{dy}{dx}\right) = 0$

$4\cdot\cancel{2}x + 5\cdot\cancel{2}y\dfrac{dy}{dx} + \cancel{2}y + \cancel{2}x\dfrac{dy}{dx} = 0$, $4x + y + \dfrac{dy}{dx}(5y+x) = 0$

$\dfrac{dy}{dx} = -\dfrac{4x+y}{x+5y}$

(c) $\left.\dfrac{dy}{dx}\right|_{(2,1)} = -\dfrac{4(2)+1}{2+5(1)} = -\dfrac{9}{7}$ $y = -\frac{9}{7}x + b$, $1 = -\frac{9}{7}(2) + b$, $b = \frac{25}{7}$

$y = -\frac{9}{7}x + \frac{25}{7}$

(d) $y = \frac{7}{9}x + b$, $1 = \frac{7}{9}(2) + b$, $b = -\frac{5}{9}$, $y = \frac{7}{9}x - \frac{5}{9}$

(e) $x + 5y = 0$, $y = -\frac{1}{5}x$, $y^2 = \frac{1}{25}x^2$, $4x^2 + 5y^2 + 2xy - 25 = 0 \Rightarrow$

$4x^2 + 5\left(\frac{1}{25}x^2\right) + 2x\left(-\frac{1}{5}x\right) - 25 = 0$, $4x^2 + \frac{1}{5}x^2 - \frac{2}{5}x^2 - 25 = 0$,

$20x^2 + x^2 - 2x^2 - 125 = 0$, $19x^2 = 125$, $x^2 = \frac{125}{19}$, $x = \pm\sqrt{\frac{125}{19}} \approx \pm 2.565$

$x \in [-2.565, 2.565]$

(f) $4x + y = 0$, $x = -\frac{1}{4}y$, $x^2 = \frac{1}{16}y^2$,

$4\left(\frac{1}{16}y^2\right) + 5y^2 + 2\left(-\frac{1}{4}y\right)y - 25 = 0$, $\frac{1}{4}y^2 + 5y^2 - \frac{1}{2}y^2 - 25 = 0$,

$y^2 + 20y^2 - 2y^2 - 100 = 0$, $19y^2 = 100$, $y^2 = \frac{100}{19}$, $y = \pm\sqrt{\frac{100}{19}} \approx \pm 2.294$

$y \in [-2.294, 2.294]$