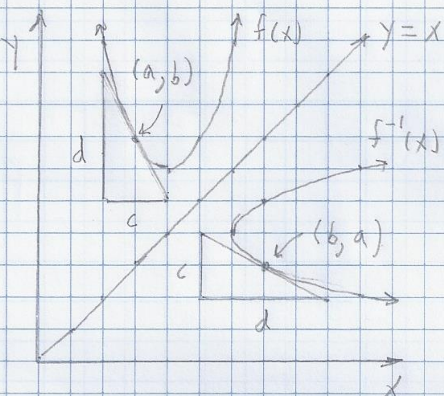
4.3. Inverse Function Theorem

Recall that the graphs of inverse functions are reflectively symmetric about the line $y = x$



$c > 0 \quad d > 0$

We see from the figure that

$$\left.\frac{df}{dx}\right|_{(a,b)} = \frac{d}{-c} \quad \text{and} \quad \left.\frac{df^{-1}}{dx}\right|_{(b,a)} = \frac{c}{-d},$$

i.e., these slopes are reciprocals.

$$\boxed{\left.\frac{df^{-1}}{dx}\right|_{(b,a)} = \frac{1}{\left.\frac{df}{dx}\right|_{(a,b)}}}$$

Example. For $f(x) = x\cos\left(\frac{2\pi x}{3}\right)$

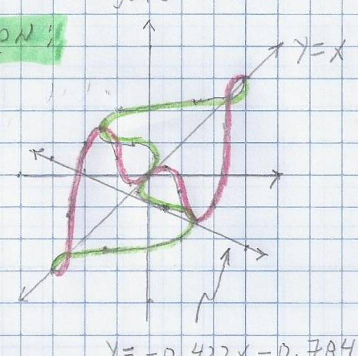
(a) Graph $y = f(x)$ on $x \in [-3, 3]$ and $y \in [-3, 3]$

(b) Graph $y = f^{-1}(x)$ by switching x and y. Also graph $y = x$.

(c) Find the equation of the line tangent to $y = f^{-1}(x)$ at $(x, y) = (0.5, -1)$. Graph the tangent line.

SOLUTION:

(a)
(b)



— $y = f(x)$
— $y = f^{-1}(x)$

(c) $f'(x) = \cos\left(\frac{2\pi x}{3}\right) + x \cdot -\sin\left(\frac{2\pi x}{3}\right) \cdot \frac{2\pi}{3}$

$f'(x) = \cos\left(\frac{2\pi x}{3}\right) - \frac{2\pi x}{3}\sin\left(\frac{2\pi x}{3}\right)$

$f'(-1) = \cos\left(-\frac{2\pi}{3}\right) + \frac{2\pi}{3}\sin\left(-\frac{2\pi}{3}\right) \approx -2.314$

$\left.\frac{df^{-1}}{dx}\right|_{(0.5,-1)} = \frac{1}{-2.314} = -0.432$, $\quad y = -0.432x + b$, $\quad -1 = -0.432(0.5) + b$

$b = -0.784$, $\quad y = -0.432x - 0.784$ ◄