4.4. Derivatives of Exponential Functions

First, we need to find the limit $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = ?$

Let $x = \ln(y+1)$, $x \to 0 \Rightarrow y \to 0$, $e^x = y+1$, $y = e^x - 1$

$$\lim_{x\to 0} \frac{e^x-1}{x} = \lim_{y\to 0} \frac{y}{\ln(y+1)} = \lim_{y\to 0} \frac{1}{\frac{1}{y}\ln(y+1)} = \lim_{y\to 0} \frac{1}{\ln(1+y)^{\frac{1}{y}}} .$$

Now, let $y = \frac{1}{z}$ or $z = \frac{1}{y}$, $y \to 0 \Rightarrow z \to \infty$,

$$\lim_{x\to 0} \frac{e^x-1}{x} = \lim_{y\to 0} \frac{1}{\ln(1+y)^{\frac{1}{y}}} = \lim_{z\to\infty} \frac{1}{\ln\left(1+\frac{1}{z}\right)^z} = \frac{1}{\ln e} = \frac{1}{1} = 1$$

$$\boxed{\lim_{z\to\infty} \left(1+\frac{1}{z}\right)^z \equiv e}$$

so, $\boxed{\lim\limits_{x\to 0} \dfrac{e^x-1}{x} = 1}$

Consider...

$$\frac{de^x}{dx} = \lim_{h\to 0} \frac{e^{x+h}-e^x}{h} = \lim_{h\to 0} \frac{e^x e^h - e^x}{h} = \lim_{h\to 0} \left(\frac{e^h-1}{h}\right)e^x = 1 \cdot e^x = e^x$$

$$\boxed{\frac{de^x}{dx} = e^x}$$

Other Bases...

$\dfrac{db^x}{dx} = ?$ $\quad b = e^{\ln b}$, $b^x = e^{x\ln b}$, $\dfrac{db^x}{dx} = e^{x\ln b} \cdot \ln b = b^x \cdot \ln b$

$$\boxed{\frac{db^x}{dx} = (\ln b)\, b^x}$$

4.4. Derivatives of Exponential Functions

Example. Calculate $f'(x)$ for

(a) $f(x) = x^2e^x - 2xe^x + 2e^x$ (b) $f(x) = \dfrac{e^x}{x^2+2}$

(c) $f(x) = x^3 4^x$ (d) $f(x) = \dfrac{3x+2}{7^x}$

SOLUTION:

(a) $f'(x) = \cancel{2xe^x} + x^2e^x - \cancel{2e^x} - \cancel{2xe^x} + \cancel{2e^x} = x^2e^x$

(b) $f(x) = \dfrac{e^x}{x^2+2} = \dfrac{u}{v}$, $\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2} = \dfrac{e^x(x^2+2) - e^x \cdot 2x}{(x^2+2)^2} =$

$= \dfrac{(x^2-2x+2)e^x}{(x^2+2)^2}$

(c) $f'(x) = 3x^2 4^x + x^3(\ln 4)4^x = \left[(\ln 4)x^3 + 3x^2\right]4^x$

(d) $f(x) = \dfrac{3x+2}{7^x} = \dfrac{u}{v}$, $\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2} = \dfrac{3 \cdot 7^x - (3x+2)(\ln 7)7^x}{(7^x)^2} =$

$= \dfrac{7^x\left[3 - (\ln 7)(3x+2)\right]}{(7^x)^2} = \dfrac{3 - (\ln 7)(3x+2)}{7^x}$

CLASS WORK

Calculate $f'(x)$ for

(a) $f(x) = \dfrac{x^2 4^x}{\ln 4} - \dfrac{2x4^x}{(\ln 4)^2} + \dfrac{2 \cdot 4^x}{(\ln 4)^3}$ (b) $f(x) = \dfrac{e^x}{4x^2 - 2x + 1}$

(c) $f(x) = (7x+3)7^x$ (d) $f(x) = \dfrac{7x-1}{3^x}$

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SOLUTIONS

(a) $$f'(x) = \frac{1}{\ln 4}\left[2x\,4^x + x^2(\ln 4)4^x\right] - \frac{1}{(\ln 4)^2}\left[2\cdot 4^x + 2x(\ln 4)4^x\right]$$

$$+ \frac{2}{(\ln 4)^3}\cdot(\ln 4)4^x = \cancel{\frac{2x\,4^x}{\ln 4}} + x^2 4^x - \cancel{\frac{2}{(\ln 4)^2}\cdot 4^x} - \cancel{\frac{2x\,4^x}{\ln 4}} + \cancel{\frac{2}{(\ln 4)^2}\cdot 4^x} =$$

$$= x^2 4^x$$

(b) $$f(x) = \frac{e^x}{4x^2-2x+1} = \frac{u}{v} \qquad \left(\frac{u}{v}\right)' = \frac{u'v-uv'}{v^2} = \frac{e^x(4x^2-2x+1) - e^x(8x-2)}{(4x^2-2x+1)^2} =$$

$$= \frac{(4x^2-10x+3)e^x}{(4x^2-2x+1)^2}$$

(c) $$f'(x) = 7\cdot 7^x + (7x+3)(\ln 7)7^x = \left[7 + (\ln 7)(7x+3)\right]\cdot 7^x$$

(d) $$f(x) = \frac{7x-1}{3^x} = \frac{u}{v} \qquad \left(\frac{u}{v}\right)' = \frac{u'v-uv'}{v^2} = \frac{7\cdot 3^x - (7x-1)(\ln 3)3^x}{(3^x)^2} =$$

$$= \frac{3^x\cdot\left[7 - (\ln 3)(7x-1)\right]}{(3^x)^2} = \frac{7-(\ln 3)(7x-1)}{3^x}$$