4.4. Derivatives Involving Absolute Values

Recall

$$\frac{d|x|}{dx} = \frac{x}{|x|}, \quad x \neq 0$$

The functions $f(x) = \ln|x|$ and $f(x) = \log_b |x|$ occur often in calculus.

$f(x) = \ln|x|$ ... $\dfrac{d \ln|x|}{dx} = \dfrac{1}{|x|} \cdot \dfrac{x}{|x|} = \dfrac{x}{x^2} = \dfrac{1}{x} \Rightarrow$

$$\boxed{\frac{d \ln|x|}{dx} = \frac{1}{x}, \quad x \neq 0}$$

$f(x) = \log_b |x|$ ... $\dfrac{d \log_b |x|}{dx} = \dfrac{1}{(\ln b)|x|} \cdot \dfrac{x}{|x|} = \dfrac{x}{(\ln b)x^2} = \dfrac{1}{(\ln b)x} \Rightarrow$

$$\boxed{\frac{d \log_b |x|}{dx} = \frac{1}{(\ln b)x}, \quad x \neq 0}$$

Examples. Calculate $f'(x)$ for

(a) $f(x) = \ln|x^2 - 3x + 7|$ (b) $f(x) = \log_7 |3x^2 - 7x + 2|$

(c) $f(x) = \ln\{|x| + \sqrt{x^2 - 1}\}$ (d) $f(x) = x^3|x|$

SOLUTIONS:

(a) $\dfrac{d}{dx} \ln|x^2 - 3x + 7| = \dfrac{1}{|x^2 - 3x + 7|} \cdot \dfrac{(x^2 - 3x + 7)}{|x^2 - 3x + 7|} \cdot (2x - 3) = \dfrac{\cancel{(x^2 - 3x + 7)}}{(x^2 - 3x + 7)^2} \cdot (2x - 3)$

$= \dfrac{2x - 3}{x^2 - 3x + 7}$

(b) $$\frac{d}{dx}\log_7|3x^2-7x+2| = \frac{1}{(\ln 7)|3x^2-7x+2|} \cdot \frac{(3x^2-7x+2)}{|3x^2-7x+2|} \cdot (6x-7) =$$

$$= \frac{6x-7}{(\ln 7)(3x^2-7x+2)}$$

(c) $$\frac{d}{dx}\left[|x|+\sqrt{x^2-1}\right] = \frac{x}{|x|} + \frac{1}{\cancel{2}}(x^2-1)^{-1/2}\cdot \cancel{2}x = \frac{x}{|x|} + \frac{x}{\sqrt{x^2-1}} =$$

$$= \frac{x}{|x|}\cdot\frac{\sqrt{x^2-1}}{\sqrt{x^2-1}} + \frac{x}{\sqrt{x^2-1}}\cdot\frac{|x|}{|x|} = \frac{x}{|x|\sqrt{x^2-1}}\left[|x|+\sqrt{x^2-1}\right]$$

$$\frac{d}{dx}\ln\left\{|x|+\sqrt{x^2-1}\right\} = \frac{1}{|x|+\sqrt{x^2-1}}\cdot\frac{d}{dx}\left[|x|+\sqrt{x^2-1}\right] =$$

$$= \frac{1}{\cancel{|x|+\sqrt{x^2-1}}}\cdot\frac{x}{|x|\sqrt{x^2-1}}\left[\cancel{|x|+\sqrt{x^2-1}}\right] = \frac{x}{|x|\sqrt{x^2-1}}$$

(d) $$f(x) = x^3|x| = \begin{bmatrix} -x^4, & -\infty < x \le 0 \\ x^4, & 0 < x < \infty \end{bmatrix} \Rightarrow$$

can tell is differentiable at $x=0$ because the derivatives from the left and right are both 0 at $x=0$

$$f'(x) = \begin{bmatrix} -4x^3, & -\infty < x \le 0 \\ 4x^3, & 0 < x < \infty \end{bmatrix} = 4|x^3|$$

CLASS WORK

Calculate $f'(x)$ for

(a) $f(x) = \ln|7x+3|$ (b) $\log_5|6x+7|$

(c) $f(x) = \ln|\sec x + \tan x|$ (d) $f(x) = x|x|$

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SOLUTIONS

(a) $\frac{d}{dx} \ln|7x+3| = \frac{1}{|7x+3|} \cdot \frac{7x+3}{|7x+3|} \cdot 7 = \frac{\cancel{(7x+3)}}{(7x+3)^{\cancel{2}}} \cdot 7 = \frac{7}{7x+3}$

(b) $\frac{d}{dx} \log_5 |6x+7| = \frac{1}{(\ln 5)|6x+7|} \cdot \frac{6x+7}{|6x+7|} \cdot 6 = \frac{\cancel{(6x+7)} \cdot 6}{(\ln 5)(6x+7)^2} = \frac{6}{(\ln 5)(6x+7)}$

(c) $\frac{d}{dx}[\sec x + \tan x] = \sec x \tan x + \sec^2 x = \sec x [\sec x + \tan x]$

$\frac{d}{dx} \ln|\sec x + \tan x| = \frac{1}{|\sec x + \tan x|} \cdot \frac{(\sec x + \tan x)}{|\sec x + \tan x|} \cdot \frac{d}{dx}[\sec x + \tan x] =$

$= \frac{(\sec x + \tan x)}{(\sec x + \tan x)^2} \cdot \sec x [\sec x + \tan x] = \sec x$

(d) $f(x) = x|x| = \begin{bmatrix} -x^2, & -\infty < x \leq 0 \\ x^2, & 0 < x < \infty \end{bmatrix} \Rightarrow f'(x) = \begin{bmatrix} -2x, & -\infty < x \leq 0 \\ 2x, & 0 < x < \infty \end{bmatrix} = 2|x|$