

4.4. Logarithmic Differentiation

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Consider

$$f(x) = x^x$$

$$\text{NOT } f'(x) = x \cdot x^{x-1} = x^x$$

Take the natural log of both sides...

$$y = x^x, \ln y = \ln x^x = x \ln x$$

$$\frac{d \ln y}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1, \quad \frac{d \ln y}{dx} = \frac{d \ln y}{dy} \frac{dy}{dx} = \frac{1}{y} \frac{dy}{dx} = \frac{1}{x^x} \frac{dx^x}{dx}$$

$$\frac{1}{x^x} \frac{dx^x}{dx} = \ln x + 1$$

$$\frac{dx^x}{dx} = (\ln x + 1) x^x$$

Example. Differentiate $f(x) = (2x)^{3x}$.

SOLUTION: $f(x) = 2^{3x} \cdot x^{3x} = 8^x \cdot x^{3x}$

$$y = x^{3x}, \ln y = \ln x^{3x} = 3x \ln x, \quad \frac{d \ln y}{dx} = 3 \left[1 \cdot \ln x + x \cdot \frac{1}{x} \right] = 3(\ln x + 1)$$

$$\frac{d \ln y}{dx} = \frac{d \ln y}{dy} \frac{dy}{dx} = \frac{1}{y} \frac{dy}{dx} = \frac{1}{x^{3x}} \frac{dx^{3x}}{dx} = 3(\ln x + 1)$$

$$\frac{dx^{3x}}{dx} = 3(\ln x + 1) x^{3x},$$

$$(2x)^{3x}$$

$$f'(x) = \frac{d 8^x}{dx} \cdot x^{3x} + 8^x \cdot \frac{dx^{3x}}{dx} = (\ln 8) 8^x \cdot x^{3x} + 8^x \cdot 3(\ln x + 1) x^{3x}$$

$$= [\ln 8 + 3(\ln x + 1)] (2x)^{3x} \leftarrow$$

CLASS WORK

Differentiate $f(x) = (4x)^{5x}$

4.4. Logarithmic Differentiation

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SOLUTION

$$f(x) = 4^{5x} \cdot x^{5x} = 1024^x \cdot x^{5x} = (4x)^{5x}$$

$$y = x^{5x} \quad \ln y = \ln x^{5x} = 5x \ln x \quad \frac{d \ln y}{dx} = 5 \left(1 \cdot \ln x + x \cdot \frac{1}{x} \right) = 5(\ln x + 1)$$

$$\frac{d \ln y}{dx} = \frac{d \ln y}{dy} \frac{dy}{dx} = \frac{1}{y} \frac{dy}{dx} = \frac{1}{x^{5x}} \frac{dx^{5x}}{dx} = 5(\ln x + 1), \quad \frac{dx^{5x}}{dx} = 5(\ln x + 1) x^{5x}$$

$$f'(x) = \frac{d 1024^x}{dx} \cdot x^{5x} + 1024^x \cdot \frac{dx^{5x}}{dx} = (\ln 1024) \cdot \underline{1024^x \cdot x^{5x}} + \underline{1024^x \cdot 5(\ln x + 1) x^{5x}}$$

$(4x)^{5x}$

$$f'(x) = [\ln 1024 + 5(\ln x + 1)] \cdot (4x)^{5x} \Leftarrow$$