## Section 5.2 Exercises

In Exercises 1–8, **(a)** state whether or not the function satisfies the hypotheses of the Mean Value Theorem on the given interval, and **(b)** if it does, find each value of $c$ in the interval $(a, b)$ that satisfies the equation

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

**1.** $f(x) = x^2 + 2x - 1$ on $[0, 1]$

**2.** $f(x) = x^{2/3}$ on $[0, 1]$

**3.** $f(x) = x^{1/3}$ on $[-1, 1]$

**4.** $f(x) = |x - 1|$ on $[0, 4]$

**5.** $f(x) = \sin^{-1} x$ on $[-1, 1]$

**6.** $f(x) = \ln(x - 1)$ on $[2, 4]$

**7.** $f(x) = \begin{cases} \cos x, & 0 \le x < \pi/2 \\ \sin x, & \pi/2 \le x \le \pi \end{cases}$ on $[0, \pi]$

**8.** $f(x) = \begin{cases} \sin^{-1} x, & -1 \le x < 1 \\ x/2 + 1, & 1 \le x \le 3 \end{cases}$ on $[-1, 3]$

In Exercises 9 and 10, the interval $a \le x \le b$ is given. Let $A = (a, f(a))$ and $B = (b, f(b))$. Write an equation for

**(a)** the secant line $AB$.

**(b)** a tangent line to $f$ in the interval $(a, b)$ that is parallel to $AB$.

**9.** $f(x) = x + \frac{1}{x}, \quad 0.5 \le x \le 2$

**10.** $f(x) = \sqrt{x - 1}, \quad 1 \le x \le 3$

**11.** ***Speeding*** A trucker handed in a ticket at a toll booth showing that in 2 h she had covered 159 mi on a toll road with speed limit 65 mph. The trucker was cited for speeding. Why?

**12.** ***Temperature Change*** It took 20 sec for the temperature to rise from 0°F to 212°F when a thermometer was taken from a freezer and placed in boiling water. Explain why at some moment in that interval the mercury was rising at exactly 10.6°F/sec.

**13.** ***Triremes*** Classical accounts tell us that a 170-oar trireme (ancient Greek or Roman warship) once covered 184 sea miles in 24 h. Explain why at some point during this feat the trireme's speed exceeded 7.5 knots (sea miles per hour).

**14.** ***Running a Marathon*** A marathoner ran the 26.2-mi New York City Marathon in 2.2 h. Show that at least twice, the marathoner was running at exactly 11 mph.

In Exercises 15–22, use analytic methods to find **(a)** the local extrema, **(b)** the intervals on which the function is increasing, and **(c)** the intervals on which the function is decreasing.

**15.** $f(x) = 5x - x^2$

**16.** $g(x) = x^2 - x - 12$

**17.** $h(x) = \frac{2}{x}$

**18.** $k(x) = \frac{1}{x^2}$

**19.** $f(x) = e^{2x}$

**20.** $f(x) = e^{-0.5x}$

**21.** $y = 4 - \sqrt{x + 2}$

**22.** $y = x^4 - 10x^2 + 9$

In Exercises 23–28, find **(a)** the local extrema, **(b)** the intervals on which the function is increasing, and **(c)** the intervals on which the function is decreasing.

**23.** $f(x) = x\sqrt{4 - x}$ **24.** $g(x) = x^{1/3}(x + 8)$

**25.** $h(x) = \dfrac{-x}{x^2 + 4}$ **26.** $k(x) = \dfrac{x}{x^2 - 4}$

**27.** $f(x) = x^3 - 2x - 2\cos x$ **28.** $g(x) = 2x + \cos x$

In Exercises 29–34, find all possible functions $f$ with the given derivative.

**29.** $f'(x) = x$ **30.** $f'(x) = 2$

**31.** $f'(x) = 3x^2 - 2x + 1$ **32.** $f'(x) = \sin x$

**33.** $f'(x) = e^x$ **34.** $f'(x) = \dfrac{1}{x - 1}, \quad x > 1$

In Exercises 35–38, find the function with the given derivative whose graph passes through the point $P$.

**35.** $f'(x) = -\dfrac{1}{x^2}, \quad x > 0, \quad P(2, 1)$

**36.** $f'(x) = \dfrac{1}{4x^{3/4}}, \quad P(1, -2)$

**37.** $f'(x) = \dfrac{1}{x + 2}, \quad x > -2, \quad P(-1, 3)$

**38.** $f'(x) = 2x + 1 - \cos x, \quad P(0, 3)$,

**Group Activity** In Exercises 39–42, sketch a graph of a differentiable function $y = f(x)$ that has the given properties.

**39.** **(a)** local minimum at $(1, 1)$, local maximum at $(3, 3)$

**(b)** local minima at $(1, 1)$ and $(3, 3)$

**(c)** local maxima at $(1, 1)$ and $(3, 3)$

**40.** $f(2) = 3, f'(2) = 0$, and

**(a)** $f'(x) > 0$ for $x < 2$, $f'(x) < 0$ for $x > 2$.

**(b)** $f'(x) < 0$ for $x < 2$, $f'(x) > 0$ for $x > 2$.

**(c)** $f'(x) < 0$ for $x \neq 2$.

**(d)** $f'(x) > 0$ for $x \neq 2$.

**41.** $f'(-1) = f'(1) = 0$, $f'(x) > 0$ on $(-1, 1)$, $f'(x) < 0$ for $x < -1$, $f'(x) > 0$ for $x > 1$.

**42.** A local minimum value that is greater than one of its local maximum values.

**43.** ***Free Fall*** On the moon, the acceleration due to gravity is $1.6 \text{ m/sec}^2$.

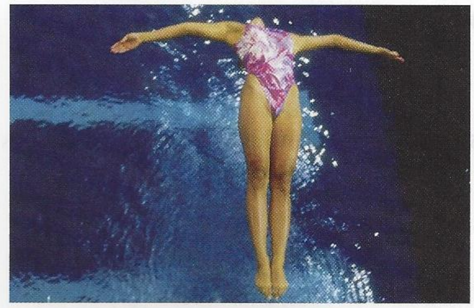
**(a)** If a rock is dropped into a crevasse, how fast will it be going just before it hits bottom 30 sec later?

**(b)** How far below the point of release is the bottom of the crevasse?

**(c)** If instead of being released from rest, the rock is thrown into the crevasse from the same point with a downward velocity of 4 m/sec, when will it hit the bottom and how fast will it be going when it does?

**44.** ***Diving*** **(a)** With what velocity will you hit the water if you step off from a 10-m diving platform?

**(b)** With what velocity will you hit the water if you dive off the platform with an upward velocity of 2 m/sec?



**45.** **Writing to Learn** The function

$$f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases}$$

is zero at $x = 0$ and at $x = 1$. Its derivative is equal to 1 at every point between 0 and 1, so $f'$ is never zero between 0 and 1, and the graph of $f$ has no tangent parallel to the chord from $(0, 0)$ to $(1, 0)$. Explain why this does not contradict the Mean Value Theorem.

**46.** **Writing to Learn** Explain why there is a zero of $y = \cos x$ between every two zeros of $y = \sin x$.

**47.** ***Unique Solution*** Assume that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Also assume that $f(a)$ and $f(b)$ have opposite signs and $f' \neq 0$ between $a$ and $b$. Show that $f(x) = 0$ exactly once between $a$ and $b$.

In Exercises 48 and 49, show that the equation has exactly one solution in the given interval. (*Hint:* See Exercise 47.)

**48.** $x^4 + 3x + 1 = 0, \quad -2 \leq x \leq -1$

**49.** $x + \ln(x + 1) = 0, \quad 0 \leq x \leq 3$

**50.** ***Parallel Tangents*** Assume that $f$ and $g$ are differentiable on $[a, b]$ and that $f(a) = g(a)$ and $f(b) = g(b)$. Show that there is at least one point between $a$ and $b$ where the tangents to the graphs of $f$ and $g$ are parallel or the same line. Illustrate with a sketch.

## Standardized Test Questions

You may use a graphing calculator to solve the following problems.

**51.** **True or False** If $f$ is differentiable and increasing on $(a, b)$, then $f'(c) > 0$ for every $c$ in $(a, b)$. Justify your answer.

**52.** **True or False** If $f$ is differentiable and $f'(c) > 0$ for every $c$ in $(a, b)$, then $f$ is increasing on $(a, b)$. Justify your answer.

**53. Multiple Choice** If $f(x) = \cos x$, then the Mean Value Theorem guarantees that somewhere between 0 and $\pi/3$, $f'(x) =$

(A) $-\frac{3}{2\pi}$ (B) $-\frac{\sqrt{3}}{2}$ (C) $-1$ (D) 0 (E) $\frac{1}{2}$

**54. Multiple Choice** On what interval is the function $g(x) = e^{x^3-6x^2+8}$ decreasing?

(A) $(-\infty, 2]$ (B) $[0, 4]$ (C) $[2, 4]$

(D) $(4, \infty)$ (E) no interval

**55. Multiple Choice** Which of the following functions is an antiderivative of $\frac{1}{\sqrt{x}}$?

(A) $-\frac{1}{\sqrt{2x^3}}$ (B) $-\frac{2}{\sqrt{x}}$ (C) $\frac{\sqrt{x}}{2}$

(D) $\sqrt{x} + 5$ (E) $2\sqrt{x} - 10$

**56. Multiple Choice** All of the following functions satisfy the conditions of the Mean Value Theorem on the interval $[-1, 1]$ *except*

(A) $\sin x$ (B) $\sin^{-1}x$ (C) $x^{5/3}$ (D) $x^{3/5}$ (E) $\frac{x}{x-2}$

## Extending the Ideas

**57. *Geometric Mean*** The **geometric mean** of two positive numbers $a$ and $b$ is $\sqrt{ab}$. Show that for $f(x) = 1/x$ on any interval $[a, b]$ of positive numbers, the value of $c$ in the conclusion of the Mean Value Theorem is $c = \sqrt{ab}$.

**58. *Arithmetic Mean*** The **arithmetic mean** of two numbers $a$ and $b$ is $(a + b)/2$. Show that for $f(x) = x^2$ on any interval $[a, b]$, the value of $c$ in the conclusion of the Mean Value Theorem is $c = (a + b)/2$.

**59. *Upper Bounds*** Show that for any numbers $a$ and $b$ $|\sin b - \sin a| \le |b - a|$.

**60. *Sign of f′*** Assume that $f$ is differentiable on $a \le x \le b$ and that $f(b) < f(a)$. Show that $f'$ is negative at some point between $a$ and $b$.

**61. *Monotonic Functions*** Show that monotonic increasing and decreasing functions are one-to-one.

**62. Writing to Learn *Proof of Rolle's Theorem*** Rolle's Theorem is the special case of the Mean Value Theorem for which $f(a) = f(b) = 0$, and the conclusion is that there is at least one point $c$ for which $f'(c) = 0$. The following steps will lead you through a proof of Rolle's Theorem.

**(a)** Since $f$ is continuous on $[a, b]$, the Extreme Value Theorem (Theorem 1, p. 194) guarantees that $f$ has both a maximum and a minimum value on this interval. Explain why, if the maximum and minimum occur only at the endpoints, then $f$ is the constant function $f(x) = 0$. It follows that either $f'(x) = 0$ for all $x$ in $(a, b)$ or there is some point $c$ between $a$ and $b$ where $f$ has a local maximum or minimum.

**(b)** Use Theorem 2 (p. 195) to finish the proof of Rolle's Theorem.

**63. Writing to Learn *Proof of the Mean Value Theorem*** The following steps will establish the Mean Value Theorem as a corollary of Rolle's Theorem. This is the proof first discovered by Ossian Bonnet in the 1860s. We assume that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$.

**(a)** Verify that

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

is the equation of the secant line through $(a, f(a))$ and $(b, f(b))$.

**(b)** Define a new function $g$,

$$g(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right).$$

Explain why $g$ is continuous on $[a, b]$, differentiable on $(a, b)$, and $g(a) = g(b) = 0$. Rolle's Theorem (Exercise 62) implies that there is at least one point $c$ between $a$ and $b$ for which $g'(c) = 0$.

**(c)** Use the definition of $g$ to show that

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Now finish the proof of the Mean Value Theorem.