Section 5.5 Exercises

In Exercises 1–6, **(a)** find the linearization $L(x)$ of $f(x)$ at $x = a$. **(b)** How accurate is the approximation $L(a + 0.1) \approx f(a + 0.1)$? See the comparisons following Example 1.

1. $f(x) = x^3 - 2x + 3, \quad a = 2$

2. $f(x) = \sqrt{x^2 + 9}, \quad a = -4$

3. $f(x) = x + \dfrac{1}{x}, \quad a = 1$

4. $f(x) = \ln(x + 1), \quad a = 0$

5. $f(x) = \tan x, \quad a = \pi$

6. $f(x) = \cos^{-1} x, \quad a = 0$

7. Show that the linearization of $f(x) = (1 + x)^k$ at $x = 0$ is $L(x) = 1 + kx$.

8. Use the linearization $(1 + x)^k \approx 1 + kx$ to approximate the following. State how accurate your approximation is.

(a) $(1.002)^{100}$ **(b)** $\sqrt[3]{1.009}$

In Exercises 9 and 10, use the linear approximation $(1 + x)^k \approx 1 + kx$ to find an approximation for the function $f(x)$ for values of x near zero.

9. (a) $f(x) = (1 - x)^6$ **(b)** $f(x) = \dfrac{2}{1 - x}$ **(c)** $f(x) = \dfrac{1}{\sqrt{1 + x}}$

10. (a) $f(x) = (4 + 3x)^{1/3}$ **(b)** $f(x) = \sqrt{2 + x^2}$

(c) $f(x) = \sqrt[3]{\left(1 - \dfrac{1}{2 + x}\right)^2}$

In Exercises 11–14, approximate the root by using a linearization centered at an appropriate nearby number.

11. $\sqrt{101}$ **12.** $\sqrt[3]{26}$

13. $\sqrt[3]{998}$ **14.** $\sqrt{80}$

In Exercises 15–22, **(a)** find dy, and **(b)** evaluate dy for the given value of x and dx.

15. $y = x^3 - 3x, \quad x = 2, \quad dx = 0.05$

16. $y = \dfrac{2x}{1 + x^2}, \quad x = -2, \quad dx = 0.1$

17. $y = x^2 \ln x, \quad x = 1, \quad dx = 0.01$

18. $y = x\sqrt{1 - x^2}, \quad x = 0, \quad dx = -0.2$

19. $y = e^{\sin x}, \quad x = \pi, \quad dx = -0.1$

20. $y = 3 \csc\left(1 - \dfrac{x}{3}\right), \quad x = 1, \quad dx = 0.1$

21. $y + xy - x = 0, \quad x = 0, \quad dx = 0.01$

22. $2y = x^2 - xy, \quad x = 2, \quad dx = -0.05$

In Exercises 23–26, find the differential.

23. $d\left(\sqrt{1 - x^2}\right)$

24. $d(e^{5x} + x^5)$

25. $d(\arctan 4x)$

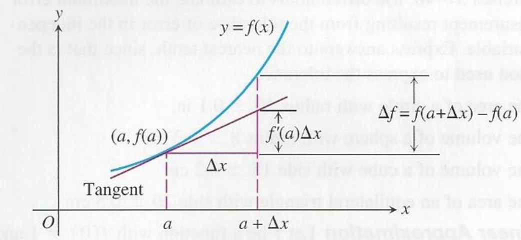
26. $d(8^x + x^8)$

In Exercises 27–30, the function f changes value when x changes from a to $a + \Delta x$. Find

(a) the true change $\Delta f = f(a + \Delta x) - f(a)$.

(b) the estimated change $f'(a)\,\Delta x$.

(c) the approximation error $|\Delta f - f'(a)\Delta x|$.



27. $f(x) = x^2 + 2x, \quad a = 0, \quad \Delta x = 0.1$

28. $f(x) = x^3 - x, \quad a = 1, \quad \Delta x = 0.1$

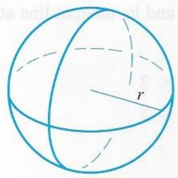
29. $f(x) = x^{-1}, \quad a = 0.5, \quad \Delta x = 0.05$

30. $f(x) = x^4, \quad a = 1, \quad \Delta x = 0.01$

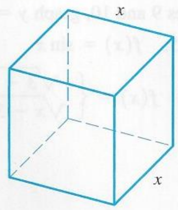
In Exercises 31–36, write a formula that estimates the given change in volume or surface area. Then use the formula to estimate the change when the independent variable changes from 10 cm to 10.05 cm.

31. ***Volume*** The change in the volume $V = (4/3)\pi r^3$ of a sphere when the radius changes from a to $a + \Delta r$

32. ***Surface Area*** The change in the surface area $S = 4\pi r^2$ of a sphere when the radius changes from a to $a + \Delta r$



$V = \frac{4}{3}\pi r^3, \quad S = 4\pi r^2$



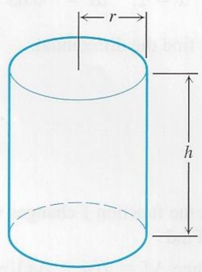
$V = x^3, \quad S = 6x^2$

33. *Volume* The change in the volume $V = x^3$ of a cube when the edge lengths change from a to $a + \Delta x$

34. *Surface Area* The change in the surface area $S = 6x^2$ of a cube when the edge lengths change from a to $a + \Delta x$

35. *Volume* The change in the volume $V = \pi r^2 h$ of a right circular cylinder when the radius changes from a to $a + \Delta r$ and the height does not change

36. *Surface Area* The change in the lateral surface area $S = 2\pi rh$ of a right circular cylinder when the height changes from a to $a + \Delta h$ and the radius does not change



$V = \pi r^2 h, \quad S = 2\pi rh$

In Exercises 37–40, use differentials to estimate the maximum error in measurement resulting from the tolerance of error in the independent variable. Express answers to the nearest tenth, since that is the precision used to express the tolerance.

37. The area of a circle with radius 10 ± 0.1 in.

38. The volume of a sphere with radius 8 ± 0.3 in.

39. The volume of a cube with side 15 ± 0.2 cm

40. The area of an equilateral triangle with side 20 ± 0.5 cm

41. *Linear Approximation* Let f be a function with $f(0) = 1$ and $f'(x) = \cos(x^2)$.

(a) Find the linearization of f at $x = 0$.

(b) Estimate the value of f at $x = 0.1$.

(c) Writing to Learn Do you think the actual value of f at $x = 0.1$ is greater than or less than the estimate in part (b)? Explain.

42. *Expanding Circle* The radius of a circle is increased from 2.00 to 2.02 m.

(a) Estimate the resulting change in area.

(b) Estimate as a percentage of the circle's original area.

43. *Growing Tree* The diameter of a tree was 10 in. During the following year, the circumference increased 2 in. About how much did the tree's diameter increase? the tree's cross-section area?

44. *Percentage Error* The edge of a cube is measured as 10 cm with an error of 1%. The cube's volume is to be calculated from this measurement. Estimate the percentage error in the volume calculation.

45. *Tolerance* About how accurately should you measure the side of a square to be sure of calculating the area to within 2% of its true value?

46. *Tolerance* **(a)** About how accurately must the interior diameter of a 10-m-high cylindrical storage tank be measured to calculate the tank's volume to within 1% of its true value?

(b) About how accurately must the tank's exterior diameter be measured to calculate the amount of paint it will take to paint the side of the tank to within 5% of the true amount?

47. *Minting Coins* A manufacturer contracts to mint coins for the federal government. The coins must weigh within 0.1% of their ideal weight, so the volume must be within 0.1% of the ideal volume. Assuming the thickness of the coins does not change, what is the percentage change in the volume of the coin that would result from a 0.1% increase in the radius?

48. *Tolerance* The height and radius of a right circular cylinder are equal, so the cylinder's volume is $V = \pi h^3$. The volume is to be calculated with an error of no more than 1% of the true value. Find approximately the greatest error that can be tolerated in the measurement of h, expressed as a percentage of h.

49. *Estimating Volume* You can estimate the volume of a sphere by measuring its circumference with a tape measure, dividing by 2π to get the radius, then using the radius in the volume formula. Find how sensitive your volume estimate is to a $1/8$-in. error in the circumference measurement by filling in the table below for spheres of the given sizes. Use the approximation for ΔV when filling in the last column.

Sphere Type	True Radius	Tape Error	Radius Error	Volume Error
Orange	2 in.	$1/8$ in.		
Melon	4 in.	$1/8$ in.		
Beach Ball	7 in.	$1/8$ in.		

50. *Estimating Surface Area* Change the heading in the last column of the table in Exercise 49 to "Surface Area Error" and find how sensitive the measure of surface area is to a $1/8$-in. error in estimating the circumference of the sphere.

51. *The Effect of Flight Maneuvers on the Heart* The amount of work done by the heart's main pumping chamber, the left ventricle, is given by the equation

$$W = PV + \frac{V\delta v^2}{2g},$$

where W is the work per unit time, P is the average blood pressure, V is the volume of blood pumped out during the unit of time, δ ("delta") is the density of the blood, v is the average velocity of the exiting blood, and g is the acceleration of gravity.

When P, V, δ, and v remain constant, W becomes a function of g, and the equation takes the simplified form

$$W = a + \frac{b}{g} \text{ (}a, b \text{ constant).}$$

As a member of NASA's medical team, you want to know how sensitive W is to apparent changes in g caused by flight maneuvers, and this depends on the initial value of g. As part of your investigation, you decide to compare the effect on W of a given change Δg on the moon, where $g = 5.2$ ft/sec^2, with the effect the same change Δg would have on Earth, where $g = 32$ ft/sec^2. Use the simplified equation above to approximate the ratio of ΔW_{moon} to ΔW_{Earth} by finding the ratio of dW_{moon}/dg to dW_{Earth}/dg.

52. *Measuring Acceleration of Gravity* When the length L of a clock pendulum is held constant by controlling its temperature, the pendulum's period T depends on the acceleration of gravity g. The period will therefore vary slightly as the clock is moved from place to place on the earth's surface, depending on the change in g. By keeping track of ΔT, we can estimate the variation in g from the equation $T = 2\pi(L/g)^{1/2}$ that relates T, g, and L.

(a) With L held constant and g as the independent variable, estimate ΔT and use it to answer parts (b) and (c).

(b) **Writing to Learn** If g increases, will T increase or decrease? Will a pendulum clock speed up or slow down? Explain.

(c) A clock with a 100-cm pendulum is moved from a location where $g = 980$ cm/sec^2 to a new location. This increases the period by $\Delta T = 0.001$ sec. Approximate Δg and use it to estimate the value of g at the new location.

Using Newton's Method on Your Calculator

Here is a nice way to get your calculator to perform the calculations in Newton's method. Try it with the function $f(x) = x^3 + 3x + 1$ from Example 13.

1. Enter the function in Y1 and its derivative in Y2.
2. On the home screen, store the initial guess into x. For example, using the initial guess in Example 13, you would type $-.3 \rightarrow X$.
3. Type X−Y1/Y2 → X and press the ENTER key over and over. Watch as the numbers converge to the zero of f. When the values stop changing, it means that your calculator has found the zero to the extent of its displayed digits, as shown in the following figure.

```
-.3→X
                  -.3
X-Y1/Y2→X
          -.322324159
         -.3221853603
         -.3221853546
         -.3221853546
```

4. Experiment with different initial guesses and repeat Steps 2 and 3.
5. Experiment with different functions and repeat Steps 1 through 3. Compare each final value you find with the value given by your calculator's built-in zero-finding feature.

In Exercises 53–56, use Newton's method to estimate all real solutions of the equation. Make your answers accurate to 6 decimal places.

53. $x^3 + x - 1 = 0$ **54.** $x^4 + x - 3 = 0$

55. $x^2 - 2x + 1 = \sin x$ **56.** $x^4 - 2 = 0$

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

57. True or False Newton's method will not find the zero of $f(x) = x/(x^2 + 1)$ if the first guess is greater than 1. Justify your answer.

58. True or False If u and v are differentiable functions, then $d(uv) = du\, dv$. Justify your answer.

59. Multiple Choice What is the linearization of $f(x) = e^x$ at $x = 1$?

(A) $y = e$ (B) $y = ex$ (C) $y = e^x$

(D) $y = x - e$ (E) $y = e(x - 1)$

60. Multiple Choice If $y = \tan x$, $x = \pi$, and $dx = 0.5$, what does dy equal?

(A) -0.25 (B) -0.5 (C) 0 (D) 0.5 (E) 0.25

61. Multiple Choice If Newton's method is used to find the zero of $f(x) = x - x^3 + 2$, what is the third estimate if the first estimate is 1?

(A) $-\frac{3}{4}$ (B) $\frac{3}{2}$ (C) $\frac{8}{5}$ (D) $\frac{18}{11}$ (E) 3

62. Multiple Choice If the linearization of $y = \sqrt[3]{x}$ at $x = 64$ is used to approximate $\sqrt[3]{66}$, what is the percentage error?

(A) 0.01% (B) 0.04% (C) 0.4% (D) 1% (E) 4%

Explorations

63. *Newton's Method* Suppose your first guess in using Newton's method is lucky in the sense that x_1 is a root of $f(x) = 0$. What happens to x_2 and later approximations?

64. *Oscillation* Show that if $h > 0$, applying Newton's method to

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ \sqrt{-x}, & x < 0 \end{cases}$$

leads to $x_2 = -h$ if $x_1 = h$, and to $x_2 = h$ if $x_1 = -h$. Draw a picture that shows what is going on.

65. *Approximations That Get Worse and Worse* Apply Newton's method to $f(x) = x^{1/3}$ with $x_1 = 1$, and calculate x_2, x_3, x_4, and x_5. Find a formula for $|x_n|$. What happens to $|x_n|$ as $n \to \infty$? Draw a picture that shows what is going on.

66. *Quadratic Approximations*

(a) Let $Q(x) = b_0 + b_1(x - a) + b_2(x - a)^2$ be a quadratic approximation to $f(x)$ at $x = a$ with the properties:

i. $Q(a) = f(a)$,
ii. $Q'(a) = f'(a)$,
iii. $Q''(a) = f''(a)$.

Determine the coefficients b_0, b_1, and b_2.

(b) Find the quadratic approximation to $f(x) = 1/(1 - x)$ at $x = 0$.

(c) Graph $f(x) = 1/(1 - x)$ and its quadratic approximation at $x = 0$. Then ZOOM IN on the two graphs at the point $(0, 1)$. Comment on what you see.

(d) Find the quadratic approximation to $g(x) = 1/x$ at $x = 1$. Graph g and its quadratic approximation together. Comment on what you see.

(e) Find the quadratic approximation to $h(x) = \sqrt{1 + x}$ at $x = 0$. Graph h and its quadratic approximation together. Comment on what you see.

(f) What are the linearizations of f, g, and h at the respective points in parts (b), (d), and (e)?

67. *Multiples of Pi* Store any number as X in your calculator. Then enter the command X − tan(X) → X and press the ENTER key repeatedly until the displayed value stops changing. The result is always an integral multiple of π. Why is this so? [*Hint:* These are zeros of the sine function.]

Extending the Ideas

68. *Formulas for Differentials* Verify the following formulas.

(a) $d(c) = 0$ (c a constant)

(b) $d(cu) = c\,du$ (c a constant)

(c) $d(u + v) = du + dv$

(d) $d(u \bullet v) = u\,dv + v\,du$

(e) $d\left(\dfrac{u}{v}\right) = \dfrac{v\,du - u\,dv}{v^2}$

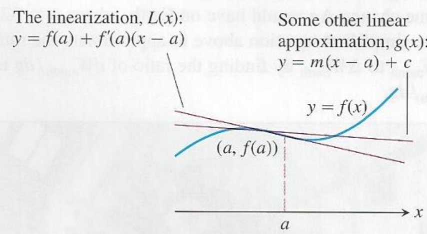
(f) $d(u^n) = nu^{n-1}\,du$

69. *The Linearization Is the Best Linear Approximation* Suppose that $y = f(x)$ is differentiable at $x = a$ and that $g(x) = m(x - a) + c$ (m and c constants). If the error $E(x) = f(x) - g(x)$ were small enough near $x = a$, we might think of using g as a linear approximation of f instead of the linearization $L(x) = f(a) + f'(a)(x - a)$. Show that if we impose on g the conditions

i. $E(a) = 0$,

ii. $\displaystyle\lim_{x \to a} \frac{E(x)}{x - a} = 0$,

then $g(x) = f(a) + f'(a)(x - a)$. Thus, the linearization gives the only linear approximation whose error is both zero at $x = a$ and negligible in comparison with $(x - a)$.



70. Writing to Learn Find the linearization of $f(x) = \sqrt{x + 1} + \sin x$ at $x = 0$. How is it related to the individual linearizations for $\sqrt{x + 1}$ and $\sin x$?

71. *Formula for Newton's Method* Derive the formula for finding the $(n + 1)$st approximation x_{n+1} from the nth approximation x_n. [*Hint:* Write the point-slope equation for the tangent line to the curve at $(x_n, f(x_n))$. See Figure 5.49 on page 245. Then set $y = 0$ and solve for x.]

72. *Bounding the Distance Between Δy and $f'(a)\,\Delta x$* We know that if $y = f(x)$, then Δy is close to $f'(a)\,\Delta x$, where $\Delta x = x - a$ and $\Delta y = f(x) - f(a)$. How close are they? In the late 1700s Joseph Louis Lagrange showed that

$$|\Delta y - f'(a)\Delta x| = \frac{1}{2}|f''(c)|(\Delta x)^2$$

for some value of c between x and a. This equality is proven in Exercise 73. While we don't know the exact value of c (if we did, this would no longer be just an approximation), we often can put an upper limit on $f''(c)$.

For the following three examples, find an upper bound on $|f''(c)|$ and use it to bound the difference $|\Delta y - f'(a)\Delta x|$ by a constant times $(\Delta x)^2$.

(a) $y = \sin x$ near $x = \pi/4$,

(b) $y = x^2$ near $x = 1$,

(c) $y = e^x$ within 0.1 unit of $x = 1$.

73. Writing to Learn The following steps can be used to prove Lagrange's result that if $y = f(x)$ has a continuous second derivative and we define g to be the difference between $\Delta y = f(x) - f(a)$ and $f'(a)\,\Delta x = f'(a)(x - a)$,

$$g(x) = (f(x) - f(a)) - f'(a)(x - a),$$

then there is some real number c between x and a such that $g(x) = \frac{1}{2}f''(c)(x - a)^2$. This proof assumes that $a < x$, but it is easily modified to handle the case $a > x$.

(a) Show that $g(a) = g'(a) = 0$ and $g''(x) = f''(x)$.

(b) Let A be the minimum value of $\frac{1}{2}g''(t)$ and B the maximum value of $\frac{1}{2}g''(t)$ for t in the interval $[a, x]$. Use the Intermediate Value Theorem to show that for any real number r between A and B, there is some value of c in $[a, x]$ for which $r = \frac{1}{2}g''(c)$.

(c) Explain why $g''(t) - 2A \geq 0$ and $g''(t) - 2B \leq 0$ for all t in the interval $[a, x]$.

(d) Using the fact that $g'(a) - 2A(a - a) = g'(a) - 2B(a - a) = 0$, together with Corollary 1 on page 204, prove that $g'(t) - 2A(t - a) \geq 0$ and $g'(t) - 2B(t - a) \leq 0$ for all t in $[a, x]$.

(e) Using the fact that $g(a) - A(a - a)^2 = g(a) - B(a - a)^2 = 0$, together with Corollary 1 on page 204, prove that $g(t) - A(t - a)^2 \geq 0$ and $g(t) - B(t - a)^2 \leq 0$ for all t in $[a, x]$.

(f) Show that $A \leq \dfrac{g(x)}{(x - a)^2} \leq B$ and therefore, by part (b), there is some value of c in $[a, x]$ for which $\dfrac{g(x)}{(x - a)^2} = \dfrac{1}{2}g''(c) = \dfrac{1}{2}f''(c)$. It follows that

$$(f(x) - f(a)) - f'(a)(x - a) = g(x) = \frac{1}{2}f''(c)(x - a)^2$$

and therefore

$$|\Delta y - f'(a)\Delta x| = \frac{1}{2}|f''(c)|(\Delta x)^2.$$