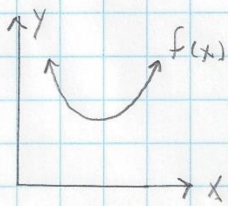
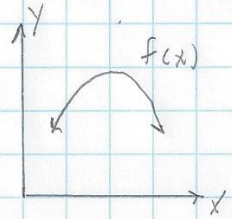
Concavity



$f''(x) > 0$

The slope is increasing

f(x) is concave up

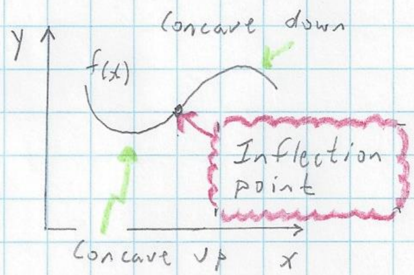


$f''(x) < 0$

The slope is decreasing

f(x) is concave down

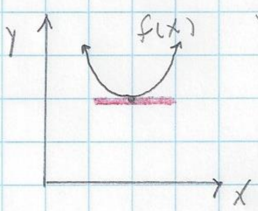
Inflection Point ≡ boundary between concave up & concave down.



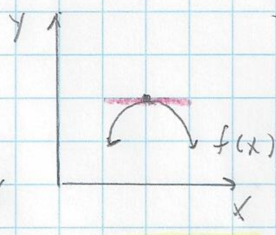
$\boxed{f''(x) = 0 \text{ is a possible inflection point}}$

Note: $f''(x) = 0$ might not be an inflection point.

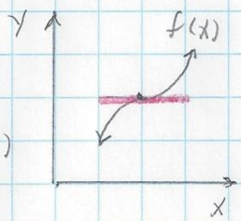
Stationary Point ≡ a point where $f'(x) = 0$

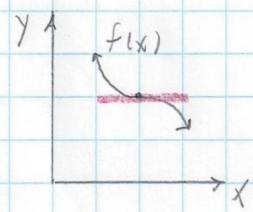


A relative minimum



A relative maximum





$\boxed{f'(x) = 0 \text{ is a possible relative extemum (maximum or minimum)}}$

Second Derivative Test for Relative Extrema

If $f'(x)=0$ and $f''(x)>0$, then x is a relative minimum of $f(x)$.

If $f'(x)=0$ and $f''(x)<0$, then x is a relative maximum of $f(x)$.

Examples. For each function $f(x)$

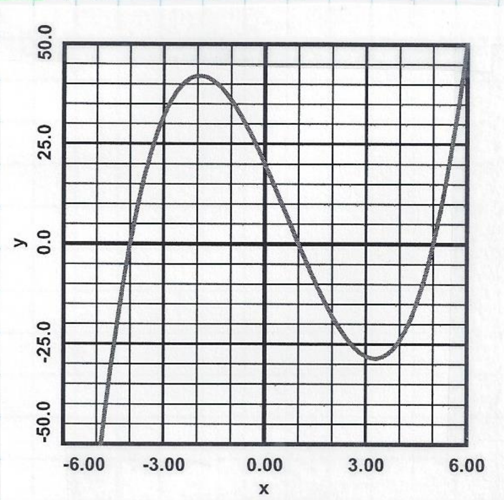
(a) Calculate $f'(x)$ and $f''(x)$

(b) Find the coordinates of the relative maxima and minima

(c) Find the coordinates of the inflection points

(d) In interval notation, state where $f(x)$ is concave up and concave down.

(1) $f(x) = x^3 - 2x^2 - 19x + 20$



$f'(x) = 3x^2 - 4x - 19$

$f''(x) = 6x - 4 = 2(3x-2)$

Possible extrema, $f'(x)=0$

$3x^2 - 4x - 19 = 0$

$x = \dfrac{4 \pm \sqrt{4^2 - 4(3)(-19)}}{2(3)} = \dfrac{4 \pm \sqrt{244}}{6}$

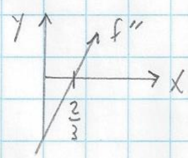
$x = -1.937$, $x = 3.270$

$f(-1.937) = 42.03$, $f''(-1.937) = -15.62 < 0$

$\Rightarrow (-1.937, 42.03)$ is rel. max

$f(3.270) = -28.55$, $f''(3.270) = 15.62 > 0 \Rightarrow (3.270, -28.55)$ is rel. min

Possible inflection points, $f''(x)=0$. $6x-4=0$, $x = \frac{4}{6} = \frac{2}{3}$, $f(\frac{2}{3}) = 6.74$



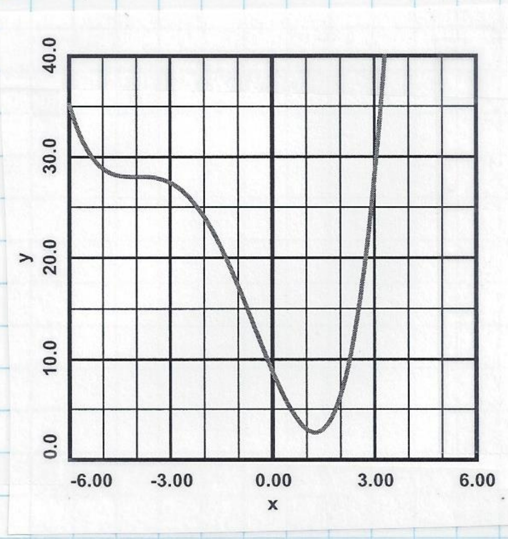
can tell that f'' switches signs at $x = \frac{2}{3}$

$\Rightarrow (\frac{2}{3}, 6.74)$ is inflection point

$f''(x) < 0$ for $x \in (-\infty, \frac{2}{3}) \Rightarrow$ concave down

$f''(x) > 0$ for $x \in (\frac{2}{3}, \infty) \Rightarrow$ concave up

(2) $f(x) = \frac{1}{10}(x+4)^3(x-3) + 28$



$f'(x) = \frac{1}{10}\left[3(x+4)^2(x-3) + (x+4)^3\right] =$

$= \frac{1}{10}(x+4)^2\left[3(x-3) + (x+4)\right] =$

$= \frac{1}{10}(x+4)^2\left[3x-9+x+4\right] =$

$= \frac{1}{10}(x+4)^2(4x-5)$ ◄

$f''(x) = \frac{1}{10}\left[2(x+4)(4x-5) + (x+4)^2 \cdot 4\right] =$

$= \frac{1}{5}(x+4)\left[(4x-5) + 2(x+4)\right] =$

$= \frac{1}{5}(x+4)(4x-5+2x+8) =$

$= \frac{1}{5}(x+4)(6x+3)$

$= \frac{3}{5}(x+4)(2x+1)$ ◄

Possible extrema, $f'(x) = 0$ $\quad (x+4)^2(4x-5) = 0$, $x = -4$, $x = 1.25$

$f(-4) = 28$, $f''(-4) = 0 \Rightarrow$ need to look at surrounding values

$f''(-4.1) = 0.432 > 0$ concave up

$f''(-3.9) = -0.408 < 0$ concave down $\Rightarrow$ 4 $\Rightarrow (-4, 28)$ is not an extreme value, but is an inflection point.

$f(1.25) = 2.68$, $f''(1.25) = 11.025 > 0 \Rightarrow (1.25, 2.68)$ is rel. min ◄

In fact, is absolute min

Possible inflection points, $f''(x) = 0$ $\quad (x+4)(2x+1) = 0$

$x = -4 \quad x = -0.5$ look at surrounding values $f''(-0.6) = -0.408$ concave down

$f''(-0.4) = 0.432$ concave up

so is inflection also

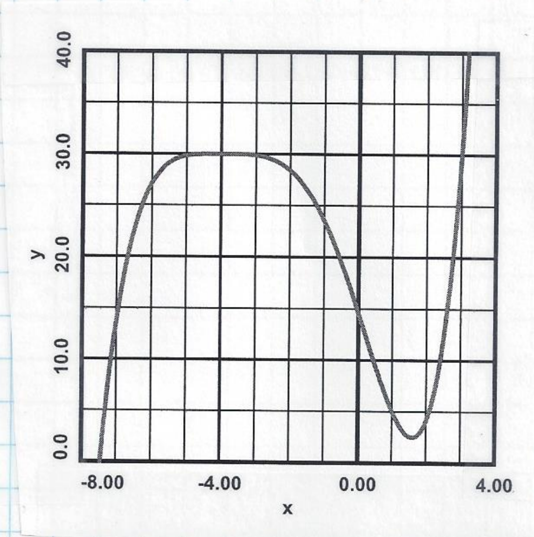
$f(-0.5) = 12.99$, inflection points are $(-4, 28)$ ◄ and $(-0.5, 12.99)$ ◄

$f''(x) > 0$ on $x \in (-\infty, -4) \cup (-0.5, \infty)$ concave up ◄

$f''(x) < 0$ on $x \in (-4, -0.5)$ concave down ◄

5.1/5.3. Analysis of Curves

(3) $f(x) = \frac{1}{50}(x+4)^4(x-3)+30$



$$f'(x) = \frac{1}{50}\left[4(x+4)^3(x-3)+(x+4)^4\right] =$$

$$= \frac{1}{50}(x+4)^3\left[4(x-3)+(x+4)\right] =$$

$$= \frac{1}{50}(x+4)^3(4x-12+x+4) =$$

$$= \frac{1}{50}(x+4)^3(5x-8)$$

$$f''(x) = \frac{1}{50}\left[3(x+4)^2(5x-8) + (x+4)^3 \cdot 5\right] =$$

$$= \frac{1}{50}(x+4)^2\left[3(5x-8)+5(x+4)\right] =$$

$$= \frac{1}{50}(x+4)^2(15x-24+5x+20) =$$

$$= \frac{1}{50}(x+4)^2(20x-4) =$$

$$= \frac{2}{25}(x+4)^2(5x-1)$$

Possible extrema, $f'(x)=0$ $\quad (x+4)^3(5x-8)=0 \quad x=-4, \; x=\frac{8}{5}=1.6$

$f(-4)=30$, $f''(-4)=0$, so need to look at surrounding values.

$f''(-4.1) = -0.0172 < 0$ concave down

$f''(-3.9) = -0.0164 < 0$ concave down

-4 $\quad f(x)$

$\Rightarrow (-4, 30)$ is rel max, and is not an inflection point

$f(1.6) = 2.46$, $f''(1.6) = 17.56$ concave up

$f(x)$ $\quad 1.6$

$\Rightarrow (1.6, 2.46)$ is rel. min.

Possible inflection points, $f''(x)=0$ $\quad (x+4)^2(5x-1)=0$

$x=-4$ (already looked at this) $\quad x=\frac{1}{5}=0.2 \quad f(0.2)=12.57$

look at surrounding values $\left.\begin{array}{l} f''(0.1) = -0.672 \\ f''(0.3) = 0.7396 \end{array}\right\}$ changes sign $\Rightarrow$

$(0.2, 12.57)$ is inflection point

$\boxed{\text{Note: } x=-4 \text{ possesses no concavity}}$

$f''(x) > 0$ on $x \in (0.2, \infty)$ concave up

$f''(x) < 0$ on $x \in (-\infty, -4) \cup (-4, 0.2)$ concave down