# 5.1/5.3. Absolute and Relative Extrema

Absolute Maximum ≡ the most maximum of all the relative maxima

Absolute Minimum ≡ the most minimum of all the relative minima

Examples. For $f(x)$ defined on the x-intervals indicated

(a) Calculate $f'(x)$ and $f''(x)$

(b) Find the coordinates of all absolute and relative extrema

Justify all answers with values of $f(x)$, $f'(x)$ and/or $f''(x)$.

(1) $f(x) = -\frac{151}{240}x^3 + \frac{451}{80}x^2 - \frac{1537}{120}x + \frac{49}{5}$, $x \in [0, 6]$.

(a) $f'(x) = -\frac{151}{80}x^2 + \frac{451}{40}x - \frac{1537}{120}$ ◂ $f''(x) = -\frac{151}{40}x + \frac{451}{40}$ ◂

(b) $f'(x) = 0 = -\frac{151}{80}x^2 + \frac{451}{40}x - \frac{1537}{120}$

$$x = \frac{-\frac{451}{40} \pm \sqrt{\left(\frac{451}{40}\right)^2 - 4\left(-\frac{151}{80}\right)\left(-\frac{1537}{120}\right)}}{2\left(-\frac{151}{80}\right)}$$

$x = 1.526$

$x = 4.448$

$f(1.526) = 1.147$ $f''(1.526) = 5.516 > 0$ $\smile f(x)$ $\Rightarrow (1.526, 1.147)$ is rel min

$f(4.448) = 8.997$ $f''(4.448) = -5.516 < 0$ $\frown f(x)$ $\Rightarrow (4.448, 8.997)$ is rel max

Look at end-points ...

$f(0) = 9.8$ $f'(0) = -12.808$ $\searrow f(x)$ $\Rightarrow (0, 9.8)$ is rel max

$f(6) = 0$ $f'(6) = -13.108$ $\nwarrow f(x)$ $\Rightarrow (6, 0)$ is rel min

$(1.526, 1.147)$ rel min ◂ $(4.448, 8.997)$ rel max ◂

$(6, 0)$ abs min ◂ $(0, 9.8)$ abs max ◂

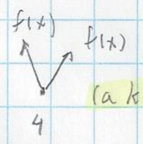
(2)

$$f(x) = \begin{cases} -\frac{53}{40}x^2 + \frac{119}{20}x + \frac{6}{5}, & 0 \le x \le 4 \\ -\frac{57}{40}x^2 + \frac{67}{4}x - \frac{202}{5}, & 4 < x \le 8 \end{cases}$$

(a) $f(4^-) = 3.8$ $f(4^+) = 3.8$ so is continuous at $x = 4$

Left: $f'(x) = -\frac{53}{20}x + \frac{119}{20}$ $f'(4-) = -4.65$

Right: $f'(x) = -\frac{57}{20}x + \frac{67}{4}$ $f'(4+) = 5.35$




(a kink) (4, 3.8) is rel min

$$f'(x) = \begin{cases} -\frac{53}{20}x + \frac{119}{20}, & 0 < x < 4 \\ -\frac{57}{20}x + \frac{67}{4}, & 4 < x \le 8 \end{cases} \qquad f''(x) = \begin{cases} -\frac{53}{20}, & 0 < x < 4 \text{ concave down} \\ -\frac{57}{20}, & 4 < x < 8 \text{ concave down} \end{cases}$$

(b) Interior points $f'(x) = 0$

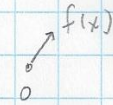
$-\frac{53}{20}x + \frac{119}{20} = 0$ $x = \frac{119}{53} = 2.245$ $f(2.245) = 7.880$ $f''(2.245) < 0 \Rightarrow$

(2.245, 7.880) is rel max

$-\frac{57}{20}x + \frac{67}{4} = 0$ $x = \frac{335}{57} = 5.877$ $f(5.877) = 8.821$ $f''(5.877) < 0 \Rightarrow$

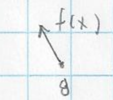
(5.877, 8.821) is rel max

Endpoints $f(0) = 1.2$ $f'(0) = 5.95$




$\Rightarrow$ (0, 1.2) is rel min

$f(8) = 2.4$ $f'(8) = -6.05$




$\Rightarrow$ (8, 2.4) is rel min

(0, 1.2) abs min ◄ (2.245, 7.880) rel max ◄
(4, 3.8) rel min ◄ (5.877, 8.821) abs max ◄
(8, 2.4) rel min ◄

## CLASS WORK

For $f(x)$ defined on the $x$-intervals indicated

(a) Calculate $f'(x)$ and $f''(x)$

(b) Find the coordinates of all absolute and relative extrema

Justify all answers with values of $f(x)$, $f'(x)$ and/or $f''(x)$.

(1) $f(x) = \frac{1}{3}x^3 - \frac{123}{40}x^2 + \frac{427}{60}x + \frac{6}{5}$, $x \in [0, 6]$

(2) $f(x) = \begin{cases} \frac{39}{80}x^2 - \frac{67}{40}x + \frac{16}{5}, & 0 \le x \le 4 \\ \frac{11}{20}x^2 - \frac{131}{20}x + \frac{217}{10}, & 4 < x \le 8 \end{cases}$

## SOLUTIONS

(1)
(a) $f'(x) = x^2 - \frac{123}{20}x + \frac{427}{60}$ $\quad f''(x) = 2x - \frac{123}{20}$

(b) $f'(x) = 0 \Rightarrow 0 = x^2 - \frac{123}{20}x + \frac{427}{60}$ $\quad x = \dfrac{\frac{123}{20} \pm \sqrt{\left(\frac{123}{20}\right)^2 - 4(1)\left(\frac{427}{60}\right)}}{2(1)}$

$x = 1.546$ $\quad f(1.546) = 6.084$ $\quad f''(1.546) = -3.059$ $\quad f(x)$ $\quad (1.546, 6.084)$ is rel max

$x = 4.604$ $\quad f(4.604) = 1.315$ $\quad f''(4.604) = 3.059$ $\quad f(x)$ $\quad (4.604, 1.315)$ is rel min

Endpoints...

$f(0) = 1.2$ $\quad f'(0) = 7.117$ $\quad$ 0 $f(x)$ $\quad (0, 1.2)$ is rel min

$f(6) = 5.2$ $\quad f'(6) = 6.217$ $\quad$ 6 $f(x)$ $\quad (6, 5.2)$ is rel max

(0, 1.2) abs min. $\quad$ (1.546, 6.084) abs max
(4.604, 1.315) rel min $\quad$ (6, 5.2) rel max

# 5.1/5.3. Absolute and Relative Extrema

(2)

(a) $f(4^-) = 4.3 \quad f(4^+) = 4.3$ so is continuous at $x=4$

Left: $f'(x) = \frac{39}{40}x - \frac{67}{40} \quad f'(4^-) = 2.225$ 4 (a kink)

Right: $f'(x) = \frac{11}{10}x - \frac{131}{20} \quad f'(4^+) = -2.15$ f(x) (4, 4.3) is rel max

$$f'(x) = \begin{cases} \frac{39}{40}x - \frac{67}{40}, & 0 \le x < 4 \\ \frac{11}{10}x - \frac{131}{20}, & 4 < x \le 8 \end{cases} \qquad f''(x) = \begin{cases} \frac{39}{40}, & 0 \le x < 4 \text{ concave up} \\ \frac{11}{10}, & 4 < x \le 8 \text{ concave up} \end{cases}$$

(b) Interior points

$f'(x) = 0 = \frac{39}{40}x - \frac{67}{40} \quad x = \frac{67}{39} = 1.718 \quad f(1.718) = 1.761 \quad f''(1.718) > 0 \Rightarrow$
(1.718, 1.761) is rel min

$f'(x) = 0 = \frac{11}{10}x - \frac{131}{20} \quad x = \frac{131}{22} = 5.955 \quad f(5.955) = 2.199 \quad f''(5.955) > 0 \Rightarrow$
(5.955, 2.199) is rel min

Endpoints...

$f(0) = 3.2 \quad f'(0) = -1.675$ 0 f(x) (0, 3.2) is rel max

$f(8) = 4.5 \quad f'(8) = 2.25$ 8 f(x) (8, 4.5) is rel max

(0, 3.2) rel max (1.718, 1.761) abs min
(4, 4.3) rel max (5.955, 2.199) rel min
(8, 4.5) abs max