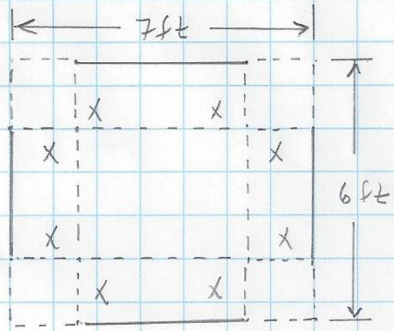
5.4. Optimization

Example #1 Four squares of side length x are cut out of the four corners of a 7ft by 6ft piece of cardboard as shown. The sides are then folded 90° to make an open-topped box of height x. Find the maximum volume of the box.



SOLUTION:

$$V = (7-2x)(6-2x)x =$$
$$= (42 - 14x - 12x + 4x^2)x = (42 - 26x + 4x^2)x =$$
$$= 2(2x^3 - 13x^2 + 21x)$$

$V'(x) = 2(6x^2 - 26x + 21)$, $V''(x) = 2(12x - 26) = 4(6x - 13)$

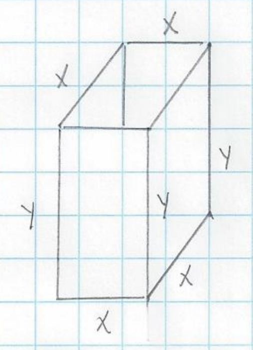
$V'(x) = 0$, $6x^2 - 26x + 21 = 0$, $x = \dfrac{26 \pm \sqrt{26^2 - 4(6)(21)}}{2(6)} = \dfrac{26 \pm \sqrt{172}}{12}$

$x = 1.074$ ✓

$x = 3.260$ ✗ the side length $(6-2x) < 0$

$V(1.074) = 20.073 \text{ ft}^3$ ◄ is max $V''(1.074) = -26.230$ ◄

Example #2. An open-topped, square-based crate has a volume of 1000 ft^3. The cost of the sides is $\$1.25/\text{ft}^2$, and the cost of the bottom is $\$3.00/\text{ft}^2$. Find the minimum cost of the crate.



SOLUTION:

$$C = 3.00x^2 + 1.25(4xy) = 3x^2 + 5xy$$

$$V = 1000 = x^2 y, \quad y = \frac{1000}{x^2}$$

$$C = 3x^2 + 5x \cdot \frac{1000}{x^2} = 3x^2 + \frac{5000}{x}$$

$$C'(x) = 6x + 5000(-1x^{-2}) = 6x - \frac{5000}{x^2}$$

$$C''(x) = 6 - 5000(-2x^{-3}) = 6 + \frac{10{,}000}{x^3}$$

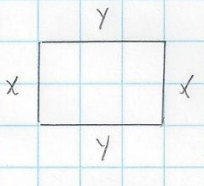
5.4. Optimization

$C'(x) = 0 = 6x - \frac{5000}{x^2}$, $\frac{5000}{x^2} = 6x$, $x^3 = \frac{5000}{6}$, $x = 9.410$

$C(9.410) = \$796.99$ is min ◄ $C''(9.410) = 18$

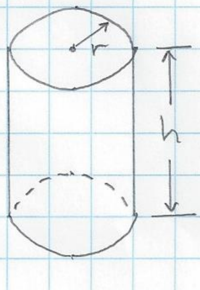
CLASS WORK

(1)



The rectangle shown has a perimeter of 100 in. Find the maximum area. What can you say about the shape of maximum area?

(2)



The can shown has volume $355\,ml = 355\,cm^3$ (= 12 fl. oz.). Find the values of r and h so that the surface area is minimized. What is the minimum surface area. Recall that, for a cylinder, the surface area is

$S = 2\pi r^2 + 2\pi r h$

and that the volume is

$V = \pi r^2 h$.

SOLUTIONS

(1) $P = 100 = 2x + 2y$ $\quad x + y = 50$ $\quad y = 50 - x$

$A = xy = x(50 - x) = -x^2 + 50x$, $A'(x) = -2x + 50$

$A'(x) = 0$, $-2x + 50 = 0$, $x = 25\,in$ ◄ $y = 50 - x = 25\,in$ ◄ It is a square ◄

$A_{max} = (25)(25) = 625\,in^2$ ◄ $\quad A''(x) = -2$

(2) $V = 355 = \pi r^2 h$, $h = \frac{355}{\pi r^2}$

$S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r\left(\frac{355}{\pi r^2}\right) = 2\pi r^2 + \frac{710}{r}$

$S'(r) = 4\pi r + 710(-1 r^{-2}) = 4\pi r - \frac{710}{r^2}$

$S''(r) = 4\pi - 710(-2r^{-3}) = 4\pi + \frac{1420}{r^3}$

$S'(r) = 0$, $4\pi r - \frac{710}{r^2} = 0$, $4\pi r = \frac{710}{r^2}$, $r^3 = \frac{355}{2\pi}$, $r = 3.837\,cm$ ◄

5.4. Optimization

$h = \frac{355}{\pi r^2} = 7.674 \text{ cm} \leftarrow$ $\quad S(3.837) = 277.55 \text{ cm}^2 \leftarrow$ $\quad S''(3.837) = 37.699$

is min