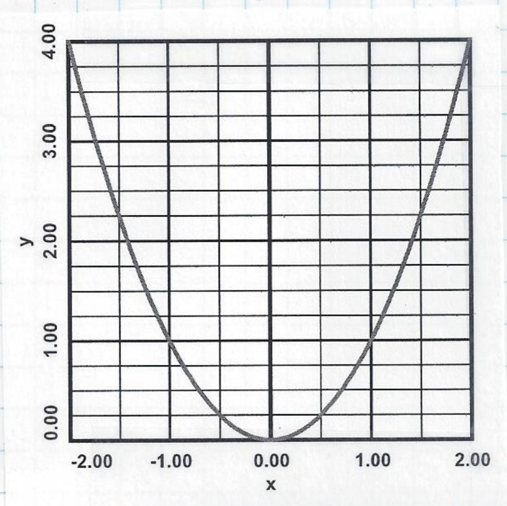
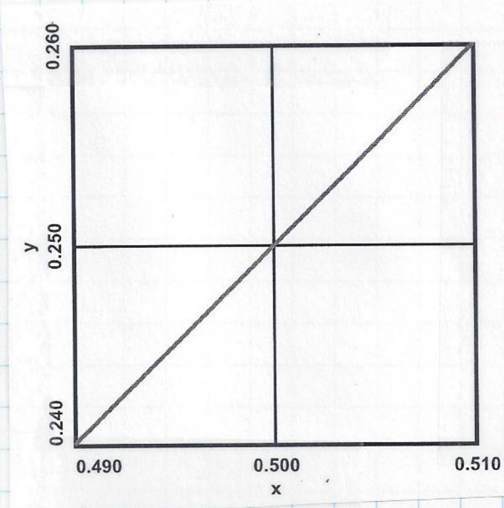
5.5. Linearization and Differentials

Consider the two graphs of $y = f(x) = x^2$.





$x \in [-2, 2]$, $y \in [0, 4]$

Global Scale

$x \in [0.49, 0.51]$, $y \in [0.24, 0.26]$

Differential Scale

Note that on the differential scale, the graph looks like a line. So, in a small neighborhood of a point (x_0, y_0) on the curve, it make sense to approximate $y = f(x)$ near $x = x_0$ by the equation of the line tangent to $y = f(x)$ at $x = x_0$.

Linearization $L(x)$. The equation of the line tangent to $y = f(x)$ at $x = x_0$ is, in point-slope form, $y - y_0 = f'(x_0)(x - x_0)$, or $y = f'(x_0)(x - x_0) + y_0$, or

$$\boxed{L(x) = f'(x_0)(x - x_0) + y_0}$$

which is called the linearization of $f(x)$ at $x = x_0$.

Example #1. Estimate the value of $f(x) = x^3 - 4x^2 - 11x + 30$ at $x = 3.1$ by using the linearization of $f(x)$ at $x = 3$. Also, calculate the relative error of the approximation.

SOLUTION:

$f'(x) = 3x^2 - 8x - 11$, $f(3) = -12$, $f'(3) = -8$, $L(x) = -8(x-3) - 12$ ◄

$L(3.1) = -12.8$ ◄ $\quad$ $\text{error} = \dfrac{L(3.1) - f(3.1)}{f(3.1)} = 0.00400 = 0.400\%$ ◄

Differential of a Function

$\frac{df}{dx} = f'(x)$ $\boxed{df = f'(x)\,dx}$

$df \equiv$ differential of f

$dx \equiv$ differential of x

$dx \equiv$ a small change in x, or an infinitesimal change in x.

Example #2. Calculate the differential of $f(x) = \frac{2x+3}{4x+5}$.

SOLUTION: $f(x) = \frac{2x+3}{4x+5} = \frac{u}{v}$, $f'(x) = \frac{u'v - uv'}{v^2} = \frac{2(4x+5) - (2x+3)\cdot 4}{(4x+5)^2} =$

$= \frac{\cancel{8x} + 10 - \cancel{8x} - 12}{(4x+5)^2} = \frac{-2}{(4x+5)^2} \Rightarrow df = \frac{-2dx}{(4x+5)^2}$

Approximate form of the Differential

$\boxed{\Delta f \approx f'(x)\,\Delta x}$

$\Delta f \equiv$ change in f

$\Delta x \equiv$ change in x

which is really just the linearization written in a different form.

Example #3. Approximate the increase in area of a circle when its radius is increased from 2 to 2.1 using the differential. Also, calculate the relative error of the approximation.

SOLUTION:

$A = A(r) = \pi r^2$, $dA = A'(r)\,dr$, $dA = (2\pi r)\,dr$

$r = 2$, $dr = 0.1$, $dA = 0.4\pi$ Exact $\Delta A = A(2.1) - A(2) = 0.41\pi$

$\text{error} = \frac{0.4\pi - 0.41\pi}{0.41\pi} = -0.0244 = -2.44\%$

CLASS WORK

(1) Estimate the value of $f(x) = x^3 + 3x^2 - 18x - 40$ at $x = -2.9$ by using the linearization of $f(x)$ at $x = -3$. Also, calculate the relative error of the approximation.

(2) Calculate the differential of $f(x) = \frac{6x+7}{8x+9}$.

(3)

(3) Estimate the increase in volume of a sphere $V=\frac{4}{3}\pi r^3$ when its radius is increased from 3 to 3.1, i.e., $r=3$ and $dr=0.1$. Also, calculate the relative error of the estimation.

SOLUTIONS

(1) $f(x)=x^3+3x^2-18x-40$, $f'(x)=3x^2+6x-18$, $f(-3)=14$, $f'(-3)=-9$,

$L(x)=-9(x+3)+14$, $L(-2.9)=13.1$ error $=\dfrac{L(-2.9)-f(-2.9)}{f(-2.9)}=0.452\%$

(2) $f(x)=\dfrac{6x+7}{8x+9}=\dfrac{u}{v}$ $f'(x)=\dfrac{u'v-uv'}{v^2}=\dfrac{6(8x+9)-(6x+7)\cdot 8}{(8x+9)^2}=$

$=\dfrac{\cancel{48x}+54-\cancel{48x}-56}{(8x+9)^2}=\dfrac{-2}{(8x+9)^2}\Rightarrow df=\dfrac{-2\,dx}{(8x+9)^2}$

(3) $V=\frac{4}{3}\pi r^3$, $V'(r)=4\pi r^2$, $dV=(4\pi r^2)dr=(4\pi\cdot 3^2)(0.1)=3.6\pi$

Exact $\Delta V=V(3.1)-V(3)=3.721\pi$, error $=\dfrac{3.6\pi-3.721\pi}{3.721}=-3.26\%$