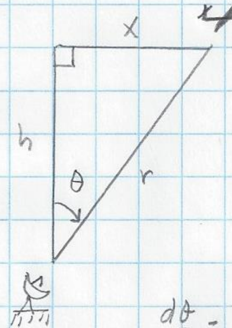
5.6. Related Rates

Example #1. A commercial airliner is cruising at $240 \frac{ft}{sec}$ at an altitude of $h = 40{,}000$ ft, and is being tracked by radar. Find $\frac{d\theta}{dt}$ when $x = 5280$ ft ($= 1$ mi).



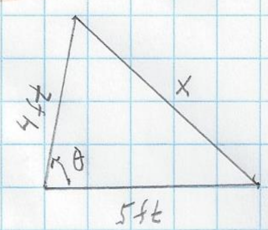
SOLUTION:

$$\tan\theta = \frac{x}{h} \qquad \frac{d\tan\theta}{dt} = \frac{d\tan\theta}{d\theta}\frac{d\theta}{dt} = \sec^2\theta \frac{d\theta}{dt} = \frac{1}{h}\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{\cos^2\theta}{h}\frac{dx}{dt} \qquad \theta = \tan^{-1}\left(\frac{x}{h}\right) = \tan^{-1}\left(\frac{5280}{40{,}000}\right) = 7.520^\circ$$

$$\frac{d\theta}{dt} = \frac{\cos^2 7.520^\circ}{40{,}000}(240) = 5.897 \times 10^{-3} \frac{rad}{sec} \leftarrow$$

Example #2. For the triangle shown, $\frac{dx}{dt} = 0.5 \frac{ft}{sec}$ when $x = 6$ft. Find $\frac{d\theta}{dt}$ at that instant.



SOLUTION: Law of cosines

$$x^2 = 4^2 + 5^2 - 2(4)(5)\cos\theta, \quad x^2 = 41 - 40\cos\theta$$

$$\frac{dx^2}{dt} = \frac{dx^2}{dx}\frac{dx}{dt} = 2x\frac{dx}{dt} = -40\frac{d\cos\theta}{dt} = -40\frac{d\cos\theta}{d\theta}\frac{d\theta}{dt} = 40\sin\theta\frac{d\theta}{dt}$$

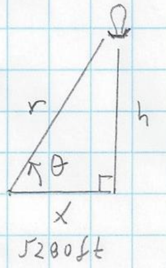
$$\frac{d\theta}{dt} = \frac{x}{20\sin\theta}\frac{dx}{dt} \qquad 6^2 = 41 - 40\cos\theta, \quad \cos\theta = \frac{1}{8}, \quad \theta = \cos^{-1}\left(\frac{1}{8}\right) = 82.82^\circ$$

$$\frac{d\theta}{dt} = \frac{6}{20\sin 82.82^\circ}(0.5) = 0.151 \frac{rad}{sec} \leftarrow$$

5.6. Related Rates

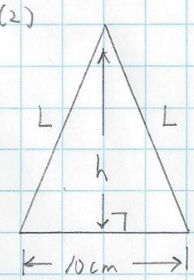
CLASS WORK

(1)



A weather balloon is rising at a rate of $\frac{dh}{dt} = 10\frac{ft}{sec}$ when $h = 50{,}000$ ft. Find $\frac{d\theta}{dt}$ at that instant.

(2)



When $L = 20$ cm, $\frac{dL}{dt} = 2\frac{cm}{sec}$. Find the rate of change in area with respect to time for the isosceles triangle at that instant.

SOLUTIONS

(1) $\tan\theta = \frac{h}{x}$ $\quad \frac{d\tan\theta}{dt} = \frac{d\tan\theta}{d\theta}\frac{d\theta}{dt} = \frac{1}{x}\frac{dh}{dt}$, $\sec^2\theta\frac{d\theta}{dt} = \frac{1}{x}\frac{dh}{dt}$, $\frac{d\theta}{dt} = \frac{\cos^2\theta}{x}\frac{dh}{dt}$,

$\tan\theta = \frac{50{,}000}{5280}$, $\theta = 83.97°$, $\frac{d\theta}{dt} = \frac{\cos^2 83.97°}{5280}(10) = 2.089 \times 10^{-5}\ \frac{rad}{sec}$

(2) $L^2 = h^2 + 5^2$, $h^2 = L^2 - 25$, $h = \sqrt{L^2 - 25}$, $A = \frac{1}{2}bh = \frac{1}{2}(10)\sqrt{L^2-25}$

$A = 5\sqrt{L^2-25}$, $\frac{dA}{dt} = 5\frac{d\sqrt{L^2-25}}{dL}\cdot\frac{dL}{dt} = 5\cdot\frac{1}{2}(L^2-25)^{-1/2}\cdot 2L\frac{dL}{dt}$

$\frac{dA}{dt} = \frac{5L}{\sqrt{L^2-25}}\frac{dL}{dt} = \frac{5(20)}{\sqrt{20^2-25}}(2) = 10.33\ \frac{cm^2}{sec}$