HW #8

pg. 208

5.2. Mean Value Theorem

(1) $f(x) = x^2 + 2x - 1$, $x \in [0, 1]$

(a) MVT applies ◄

(b) $f'(x) = 2x + 2 = 2(x+1)$ $\quad \frac{f(1) - f(0)}{1 - 0} = 3$, $3 = 2(c+1)$, $1.5 = c + 1$, $c = 0.5$ ◄

(4) $f(x) = |x - 1|$, $x \in [0, 4]$

(a) f is not differentiable at $x = 1 \Rightarrow$ MVT does not apply ◄

(8) $f(x) = \begin{cases} \sin^{-1} x, & -1 \le x < 1 \\ \frac{1}{2}x + 1, & 1 \le x \le 3 \end{cases}$ $\quad f(1^-) = \frac{\pi}{2}$, $f(1^+) = 1.5$

(a) f is not continuous at $x = 1 \Rightarrow$ MVT does not apply ◄

(11) average speed $= \frac{159 \text{ mi}}{2 \text{ hr}} = 79.5 \frac{\text{mi}}{\text{hr}}$, which she was driving at some point (due to the MVT) so she was speeding $(79.5 > 65)$ ◄

Supplemental:

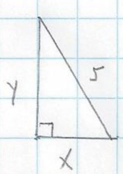
(1) $f(x) = x^3 - 9x^2 + 18x + 12$, $x \in [-2, 10]$ $\quad \frac{f(10) - f(-2)}{10 - -2} = 30$

$f'(x) = 3x^2 - 18x + 18$, $\quad 30 = 3c^2 - 18c + 18$, $\quad 3c^2 - 18c - 12 = 0$

$$c = \frac{18 \pm \sqrt{18^2 - 4(3)(-12)}}{2(3)} = \frac{18 \pm \sqrt{468}}{6} \quad c = -0.6056 ◄ \quad c = 6.6056 ◄$$

pg. 231 5.4. Optimization

(2)



$A = \frac{1}{2}xy$, $x^2 + y^2 = 5^2$, $y = \sqrt{25 - x^2}$, $A = \frac{1}{2}x\sqrt{25 - x^2}$

$$A'(x) = \frac{1}{2}\left[1 \cdot \sqrt{25-x^2} \cdot \frac{\sqrt{25-x^2}}{\sqrt{25-x^2}} + x \cdot \cancel{\frac{1}{2}} \frac{1}{\sqrt{25-x^2}} \cdot -\cancel{2}x\right]$$

$$= \frac{1}{2}\left[\frac{25 - x^2 - x^2}{\sqrt{25-x^2}}\right] = \frac{25 - 2x^2}{2\sqrt{25-x^2}} = \frac{u}{v}$$

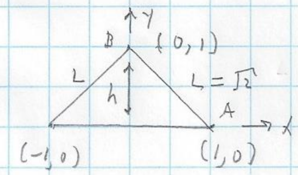
$$A''(x) = \frac{u'v - uv'}{v^2} = \frac{-4x \cdot 2\sqrt{25-x^2} \cdot \frac{\sqrt{25-x^2}}{\sqrt{25-x^2}} - (25 - 2x^2) \cdot \cancel{2}\cancel{\frac{1}{2}} \frac{1}{\sqrt{25-x^2}} \cdot -2x}{(\sqrt{25-x^2})^2} =$$

$$= \frac{-8x(25 - x^2) + 2x(25 - 2x^2)}{(\sqrt{25-x^2})^3} = \frac{2x(2x^2 - 75)}{(\sqrt{25-x^2})^3}$$

$A'(x)=0$, $25-2x^2=0$, $x=\sqrt{\frac{25}{2}}=3.536$ cm $A''(3.536)=-8$ so max.

$A(3.536)=6.25$ cm^2 $y=\sqrt{25-x^2}=3.536$ cm

(5)



$L^2+L^2=2^2$ $2L^2=4$ $L=\sqrt{2}$ $h^2+1^2=(\sqrt{2})^2$ $h^2+1=2$

$h=1$. Line AB: $y=-x+1=1-x$

$A=(\text{base})(\text{height})=(2x)(1-x)=-2x^2+2x$

$A'(x)=-4x+2$, $A'(x)=0=-4x+2$, $x=0.5$ $y=0.5$

$A=(0.5)^2=0.25$ $A''=-4$ so max.

(9)

$x+2y=800$ $A=xy$ $x=800-2y$ $A=(800-2y)y=-2y^2+800y$

$A'(y)=-4y+800=0$ $y=200$ m $x=400$ m

$A=(200)(400)=80{,}000$ m^2 $A''=-4$ so max.

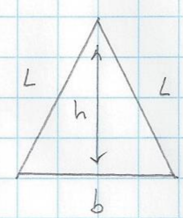
(23) $p\equiv$ profit $p=r-c$ $p(x)=8\sqrt{x}-2x^2=8\cdot\frac{1}{2\sqrt{x}}-4x=4\left(\frac{1}{\sqrt{x}}-x\right)$,

$p''(x)=4\left(-\frac{1}{2}x^{-3/2}-1\right)=-4\left(\frac{1}{2\sqrt{x^3}}+1\right)$, $p'(x)=0=\frac{1}{\sqrt{x}}-x$, $x\sqrt{x}=1$

$x=1$ $p''=-6$ so max.

Supplemental:

(2)



$2L+b=30$, $b=30-2L$, $h^2+\left(\frac{b}{2}\right)^2=L^2$, $L^2=h^2+\frac{b^2}{4}$

$h^2=\frac{1}{4}(4L^2-b^2)$, $b^2=900-120L+4L^2$

$h^2=\frac{1}{4}(4L^2-900+120L-4L^2)=30L-225$, $h=\sqrt{30L-225}$

$A=\frac{1}{2}bh=\frac{1}{2}(30-2L)\sqrt{30L-225}=(15-L)\sqrt{30L-225}$

$A'(L)=-1\cdot\sqrt{30L-225}\cdot\frac{\sqrt{30L-225}}{\sqrt{30L-225}}+(15-L)\cdot\frac{1}{2}\frac{1}{\sqrt{30L-225}}\cdot 30$

$=\frac{-(30L-225)+15(15-L)}{\sqrt{30L-225}}=\frac{450-45L}{\sqrt{30L-225}}=\frac{45(10-L)}{\sqrt{30L-225}}=\frac{u}{v}$

$$A''(L) = \frac{u'v-uv'}{v^2} = \frac{-45\sqrt{30L-225}\cdot\frac{\sqrt{30L-225}}{\sqrt{30L-225}} - 45(10-L)\cdot\frac{1}{2}\frac{1}{\sqrt{30L-225}}\cdot 30}{(\sqrt{30L-225})^2} =$$

$$= \frac{-45\left[30L-225+15(10-L)\right]}{(\sqrt{30L-225})^3} = \frac{675(5-L)}{(\sqrt{30L-225})^3}$$

$A'(L)=0$, $10-L=0$ $L=10$ ft ◄ $b=30-2L=10$ ft ◄ so, it is an equilateral triangle ◄

$h=\sqrt{30L-225}=5\sqrt{3}$ ◄ $A=\frac{1}{2}bh = 25\sqrt{3}$ ft^2 = 43.30 ft^2 ◄

$A''(10) = -5.196$ so max.

pg. 247 5.5. Linearization and Differentials

(1) $f(x)=x^3-2x+3$, $f'(x)=3x^2-2$, $f(2)=7$, $f'(2)=10$, $L(x)=10(x-2)+7$ ◄

$L(2.1)=8$ error $= \frac{L(2.1)-f(2.1)}{f(2.1)} = -0.757\%$ ◄

(5) $f(x)=\tan x$, $f'(x)=\sec^2 x$, $f(\pi)=0$, $f'(\pi)=1$, $L(x)=x-\pi$ ◄

$L(\pi+0.1)=0.1$ error $= \frac{L(\pi+0.1)-f(\pi+0.1)}{f(\pi+0.1)} = -0.334\%$ ◄

(13) $\sqrt[3]{x} = x^{1/3} = f(x)$, $f'(x)=\frac{1}{3}x^{-2/3} = \frac{1}{3\cdot\sqrt[3]{x^2}}$ center at $x=1000$

$f(1000)=10$, $f'(1000)=\frac{1}{300}$, $L(x)=\frac{1}{300}(x-1000)+10$ ◄

$L(998)=9.99\overline{3}$ error $= \frac{L(998)-f(998)}{f(998)} = 4.453\times10^{-5}\%$ ◄

(16) $y=\frac{2x}{1+x^2}=\frac{u}{v}$, $y'=\frac{u'v-uv'}{v^2} = \frac{2(1+x^2)-2x\cdot 2x}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2}$, $dy=y'(x)dx$ ◄

$dx=0.1$, $x=-2$, $dy=y'(-2)(0.1) = -0.024$ ◄ Exact Δy is

$\Delta y = y(-1.9)-y(-2) = -0.024295$ error $= \frac{dy-\Delta y}{\Delta y} = -1.214\%$ ◄

(17) $y=x^2\ln x$, $y'=2x\ln x + x^2\cdot\frac{1}{x} = x+2x\ln x$, $dy=y'(x)dx$

$dx=0.01$, $x=1$, $dy=f'(1)(0.01)=0.01$ ◄, Exact Δy is

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$\Delta y = f(1.01) - f(1) = 0.01015 \qquad \text{error} = \dfrac{dy - \Delta y}{\Delta y} = -1.481\%$

(25) $f = \tan^{-1} 4x \,,\; f'(x) = \dfrac{1}{1+(4x)^2} \cdot 4 = \dfrac{4}{1+16x^2} \,,\; df = f'(x)\,dx = \dfrac{4\,dx}{1+16x^2}$

(26) $f = 9^x + 8x^2 \,,\; f'(x) = (\ln 9)9^x + 16x \,,\; df = f'(x)\,dx = \left[(\ln 9)9^x + 16x\right]dx$

pg. 257 5.6 Related Rates

(5) $s = \sqrt{x^2+y^2+z^2} \,,\; \dfrac{ds}{dt} = \dfrac{1}{2\sqrt{x^2+y^2+z^2}}\left[\dfrac{dx^2}{dt} + \dfrac{dy^2}{dt} + \dfrac{dz^2}{dt}\right] = \dfrac{1}{2s}\left[2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} + 2z\dfrac{dz}{dt}\right] =$

$= \dfrac{1}{s}\left[x\dfrac{dx}{dt} + y\dfrac{dy}{dt} + z\dfrac{dz}{dt}\right]$

(9)

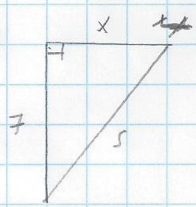
(a) $A = lw \,,\; \dfrac{dA}{dt} = \dfrac{dl}{dt}w + l\dfrac{dw}{dt} = (-2)(5) + (12)(2) = 14\,\dfrac{cm^2}{sec}$

(b) $p = 2(l+w) \,,\; \dfrac{dp}{dt} = 2\left(\dfrac{dl}{dt} + \dfrac{dw}{dt}\right) = 2(-2+2) = 0\,\dfrac{cm}{sec}$

(c) $d = \sqrt{l^2+w^2} \,,\; \dfrac{dd}{dt} = \dfrac{1}{2\sqrt{l^2+w^2}}\left[\dfrac{dl^2}{dt} + \dfrac{dw^2}{dt}\right] = \dfrac{1}{2d}\left[2l\dfrac{dl}{dt} + 2w\dfrac{dw}{dt}\right] =$

$= \dfrac{1}{d}\left[l\dfrac{dl}{dt} + w\dfrac{dw}{dt}\right] = \dfrac{1}{13}\left[(12)(-2) + (5)(2)\right] = -\dfrac{14}{13}\,\dfrac{cm}{sec}$

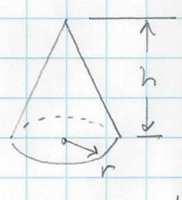
(13)



$s^2 = x^2 + 7^2 \qquad 2s\dfrac{ds}{dt} = 2x\dfrac{dx}{dt} \qquad \dfrac{dx}{dt} = \dfrac{s}{x}\dfrac{ds}{dt}$

$x^2 = s^2 - 7^2 \,,\; x = \sqrt{10^2 - 7^2} = \sqrt{51} \,,\; \dfrac{dx}{dt} = \dfrac{10}{\sqrt{51}}(300) = 420.08 \text{ mph}$

(16)



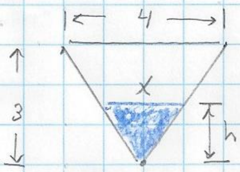
$V = \frac{1}{3}\pi r^2 h \,,\; h = \frac{3}{8}d = \frac{3}{8}(2r) = \frac{3}{4}r \,,\; V = \frac{1}{3}\pi r^2\left(\frac{3}{4}r\right) = \frac{1}{4}\pi r^3$

$\dfrac{dV}{dt} = \dfrac{1}{4}\pi \cdot 3r^2\dfrac{dr}{dt} = \dfrac{3}{4}\pi r^2\dfrac{dr}{dt} \,,\; r = \dfrac{4}{3}h = \dfrac{4}{3}(4) = \dfrac{16}{3} \,,$

$10 = \dfrac{3}{4}\pi\left(\dfrac{16}{3}\right)^2\dfrac{dr}{dt} = \dfrac{64\pi}{3}\dfrac{dr}{dt} \,,\; \dfrac{dr}{dt} = \dfrac{15}{32\pi} = 0.1492\,\dfrac{m}{s}$

$\dfrac{dh}{dt} = \dfrac{3}{4}\dfrac{dr}{dt} = 0.1119\,\dfrac{m}{s}$

(20)

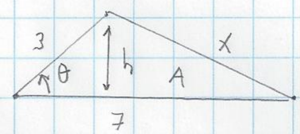


$\frac{x}{4} = \frac{h}{3}$ $\quad x = \frac{4}{3}h$ $\quad V = \frac{1}{2}xh \cdot 15 = \frac{15}{2}xh = \frac{15}{2}\left(\frac{4}{3}h\right)h = 10h^2$

$\frac{dV}{dt} = 10 \cdot 2h\frac{dh}{dt} = 20h\frac{dh}{dt}$, $\quad 2.5 = 20(2)\frac{dh}{dt}$, $\quad \frac{dh}{dt} = \frac{1}{16}\frac{ft}{min}$

Supplemental:

(3)



$\sin\theta = \frac{h}{3}$, $\quad h = 3\sin\theta$, $\quad A = \frac{1}{2}bh = \frac{1}{2}\cdot 7 \cdot 3\sin\theta$

$A = \frac{21}{2}\sin\theta$, $\quad \frac{dA}{dt} = \frac{21}{2}\cos\theta\frac{d\theta}{dt}$, $\quad \frac{d\theta}{dt} = \frac{2}{21\cos\theta}\frac{dA}{dt}$.

(a) $A = 7 = \frac{21}{2}\sin\theta$, $\quad \sin\theta = \frac{2}{3}$, $\quad \theta = 41.81°$, $\quad \frac{d\theta}{dt} = \frac{2}{21\cos 41.81°}(0.5) = 0.0639\frac{rad}{min}$

$\frac{0.0639\,rad}{min}\left(\frac{180\,deg}{\pi\,rad}\right) = 3.66\frac{deg}{min}$ $\qquad$ (b) $x^2 = 3^2 + 7^2 - 2(3)(7)\cos\theta$

$x^2 = 58 - 42\cos\theta$, $\quad 2x\frac{dx}{dt} = -42 \cdot -\sin\theta\frac{d\theta}{dt}$, $\quad \frac{dx}{dt} = \frac{21\sin\theta}{x}\frac{d\theta}{dt}$,

$x^2 = 58 - 42\cos 41.81°$, $\quad x = 5.1667$, $\quad \frac{dx}{dt} = \frac{21\sin 41.81°}{5.1667}(0.0639) = 0.1731\frac{ft}{min}$