Section 6.3 Exercises

The exercises in this section are designed to reinforce your understanding of the definite integral from the algebraic and geometric points of view. For this reason, you should not use the numerical integration capability of your calculator (NINT) except perhaps to support an answer.

1. Suppose that f and g are continuous functions and that

$$\int_1^2 f(x)\,dx = -4, \quad \int_1^5 f(x)\,dx = 6, \quad \int_1^5 g(x)\,dx = 8.$$

Use the rules in Table 6.3 to find each integral.

(a) $\int_2^2 g(x)\,dx$ **(b)** $\int_5^1 g(x)\,dx$

(c) $\int_1^2 3f(x)\,dx$ **(d)** $\int_2^5 f(x)\,dx$

(e) $\int_1^5 [f(x) - g(x)]\,dx$ **(f)** $\int_1^5 [4f(x) - g(x)]\,dx$

2. Suppose that f and h are continuous functions and that

$$\int_1^9 f(x)\,dx = -1, \quad \int_7^9 f(x)\,dx = 5, \quad \int_7^9 h(x)\,dx = 4.$$

Use the rules in Table 6.3 to find each integral.

(a) $\int_1^9 -2f(x)\,dx$ **(b)** $\int_7^9 [f(x) + h(x)]\,dx$

(c) $\int_7^9 [2f(x) - 3h(x)]\,dx$ **(d)** $\int_9^1 f(x)\,dx$

(e) $\int_1^7 f(x)\,dx$ **(f)** $\int_9^7 [h(x) - f(x)]\,dx$

3. Suppose that $\int_1^2 f(x)\,dx = 5$. Find each integral.

(a) $\int_1^2 f(u)\,du$ **(b)** $\int_1^2 \sqrt{3}\,f(z)\,dz$

(c) $\int_2^1 f(t)\,dt$ **(d)** $\int_1^2 [-f(x)]\,dx$

4. Suppose that $\int_{-3}^0 g(t)\,dt = \sqrt{2}$. Find each integral.

(a) $\int_0^{-3} g(t)\,dt$ **(b)** $\int_{-3}^0 g(u)\,du$

(c) $\int_{-3}^0 [-g(x)]\,dx$ **(d)** $\int_{-3}^0 \frac{g(r)}{\sqrt{2}}\,dr$

5. Suppose that f is continuous and that

$$\int_0^3 f(z)\,dz = 3 \quad \text{and} \quad \int_0^4 f(z)\,dz = 7.$$

Find each integral.

(a) $\int_3^4 f(z)\,dz$ **(b)** $\int_4^3 f(t)\,dt$

6. Suppose that h is continuous and that

$$\int_{-1}^1 h(r)\,dr = 0 \quad \text{and} \quad \int_{-1}^3 h(r)\,dr = 6.$$

Find each integral.

(a) $\int_1^3 h(r)\,dr$ **(b)** $-\int_3^1 h(u)\,du$

7. Show that the value of $\int_0^1 \sin(x^2)\,dx$ cannot possibly be 2.

8. Show that the value of $\int_0^1 \sqrt{x+8}\,dx$ lies between $2\sqrt{2} \approx 2.8$ and 3.

9. ***Integrals of Nonnegative Functions*** Use the Max-Min Inequality to show that if f is integrable then

$$f(x) \geq 0 \text{ on } [a, b] \Rightarrow \int_a^b f(x)\,dx \geq 0.$$

10. ***Integrals of Nonpositive Functions*** Show that if f is integrable then

$$f(x) \leq 0 \text{ on } [a, b] \Rightarrow \int_a^b f(x)\,dx \leq 0.$$

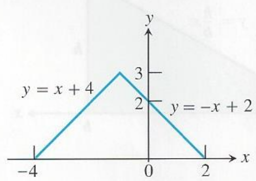
In Exercises 11–14, use NINT to find the average value of the function on the interval. At what point(s) in the interval does the function assume its average value?

11. $y = x^2 - 1, \quad [0, \sqrt{3}]$ **12.** $y = -\dfrac{x^2}{2}, \quad [0, 3]$

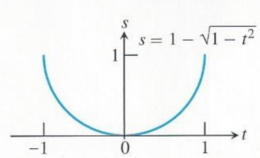
13. $y = -3x^2 - 1, \quad [0, 1]$ **14.** $y = (x-1)^2, \quad [0, 3]$

In Exercises 15–18, find the average value of the function on the interval without integrating, by appealing to the geometry of the region between the graph and the x-axis.

15. $f(x) = \begin{cases} x + 4, & -4 \leq x \leq -1, \\ -x + 2, & -1 < x \leq 2, \end{cases}$ on $[-4, 2]$



16. $f(t) = 1 - \sqrt{1 - t^2}, \quad [-1, 1]$



17. $f(t) = \sin t, \quad [0, 2\pi]$

18. $f(\theta) = \tan\theta, \quad \left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$

In Exercises 19–30, interpret the integrand as the rate of change of a quantity and evaluate the integral using the antiderivative of the quantity, as in Example 4.

19. $\int_\pi^{2\pi} \sin x\,dx$ **20.** $\int_0^{\pi/2} \cos x\,dx$

21. $\int_0^1 e^x\,dx$ **22.** $\int_0^{\pi/4} \sec^2 x\,dx$

23. $\int_1^4 2x\,dx$ **24.** $\int_{-1}^2 3x^2\,dx$

25. $\int_{-2}^6 5\,dx$ **26.** $\int_3^7 8\,dx$

27. $\int_{-1}^1 \dfrac{1}{1+x^2}\,dx$ **28.** $\int_0^{1/2} \dfrac{1}{\sqrt{1-x^2}}\,dx$

29. $\int_1^e \dfrac{1}{x}\,dx$ **30.** $\int_1^4 -x^{-2}\,dx$

In Exercises 31–36, find the average value of the function on the interval, using antiderivatives to compute the integral.

31. $y = \sin x, \quad [0, \pi]$ **32.** $y = \dfrac{1}{x}, \quad [e, 2e]$

33. $y = \sec^2 x, \quad \left[0, \dfrac{\pi}{4}\right]$ **34.** $y = \dfrac{1}{1+x^2}, \quad [0, 1]$

35. $y = 3x^2 + 2x, \quad [-1, 2]$ **36.** $y = \sec x \tan x, \quad \left[0, \dfrac{\pi}{3}\right]$

37. Group Activity Use the Max-Min Inequality to find upper and lower bounds for the value of

$$\int_0^1 \frac{1}{1+x^4}\,dx.$$

38. Group Activity *(Continuation of Exercise 37)* Use the Max-Min Inequality to find upper and lower bounds for the values of

$$\int_0^{0.5} \frac{1}{1+x^4}\,dx \quad \text{and} \quad \int_{0.5}^1 \frac{1}{1+x^4}\,dx.$$

Add these to arrive at an improved estimate for

$$\int_0^1 \frac{1}{1+x^4}\,dx.$$

39. Writing to Learn If $av(f)$ really is a typical value of the integrable function $f(x)$ on $[a, b]$, then the number $av(f)$ should have the same integral over $[a, b]$ that f does. Does it? That is, does

$$\int_a^b av(f)\,dx = \int_a^b f(x)\,dx?$$

Give reasons for your answer.

40. Writing to Learn A driver averaged 30 mph on a 150-mile trip and then returned over the same 150 miles at the rate of 50 mph. He figured that his average speed was 40 mph for the entire trip.

(a) What was his total distance traveled?

(b) What was his total time spent for the trip?

(c) What was his average speed for the trip?

(d) Explain the error in the driver's reasoning.

41. Writing to Learn A dam released 1000 m^3 of water at 10 m^3/min and then released another 1000 m^3 at 20 m^3/min. What was the average rate at which the water was released? Give reasons for your answer.

42. Use the inequality $\sin x \le x$, which holds for $x \ge 0$, to find an upper bound for the value of $\int_0^1 \sin x\,dx$.

43. The inequality $\sec x \ge 1 + (x^2/2)$ holds on $(-\pi/2, \pi/2)$. Use it to find a lower bound for the value of $\int_0^1 \sec x\,dx$.

44. Show that the average value of a linear function $L(x)$ on $[a, b]$ is

$$\frac{L(a) + L(b)}{2}.$$

(*Caution:* This simple formula for average value does *not* work for functions in general!)

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

45. True or False The average value of a function f on $[a, b]$ always lies between $f(a)$ and $f(b)$. Justify your answer.

46. True or False If $\int_a^b f(x)\,dx = 0$, then $f(a) = f(b)$. Justify your answer.

47. Multiple Choice If $\int_3^7 f(x)\,dx = 5$ and $\int_3^7 g(x)\,dx = 3$, then all of the following must be true *except*

(A) $\displaystyle\int_3^7 f(x)g(x)\,dx = 15$

(B) $\displaystyle\int_3^7 [f(x) + g(x)]\,dx = 8$

(C) $\displaystyle\int_3^7 2f(x)\,dx = 10$

(D) $\displaystyle\int_3^7 [f(x) - g(x)]\,dx = 2$

(E) $\displaystyle\int_7^3 [g(x) - f(x)]\,dx = 2$

48. Multiple Choice If $\int_2^5 f(x)\,dx = 12$ and $\int_5^8 f(x)\,dx = 4$, then all of the following must be true *except*

(A) $\displaystyle\int_2^8 f(x)\,dx = 16$

(B) $\displaystyle\int_2^5 f(x)\,dx - \int_5^8 3f(x)\,dx = 0$

(C) $\displaystyle\int_5^2 f(x)\,dx = -12$

(D) $\displaystyle\int_{-5}^{-8} f(x)\,dx = -4$

(E) $\displaystyle\int_2^6 f(x)\,dx + \int_6^8 f(x)\,dx = 16$

49. Multiple Choice What is the average value of the cosine function on the interval $[1, 5]$?

(A) -0.990 (B) -0.450 (C) -0.128

(D) 0.412 (E) 0.998

50. Multiple Choice If the average value of the function f on the interval $[a, b]$ is 10, then $\int_a^b f(x)\,dx =$

(A) $\dfrac{10}{b-a}$ (B) $\dfrac{f(a)+f(b)}{10}$ (C) $10b - 10a$

(D) $\dfrac{b-a}{10}$ (E) $\dfrac{f(b)+f(a)}{20}$

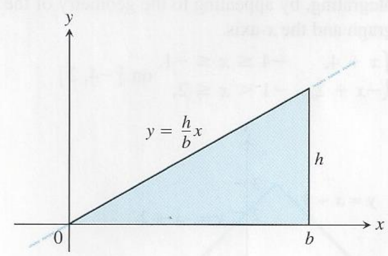
Exploration

51. *Comparing Area Formulas* Consider the region in the first quadrant under the curve $y = (h/b)x$ from $x = 0$ to $x = b$ (see figure).

(a) Use a geometry formula to calculate the area of the region.

(b) Find all antiderivatives of y.

(c) Use an antiderivative of y to evaluate $\int_0^b y(x)\,dx$.



Extending the Ideas

52. *Graphing Calculator Challenge* If $k > 1$, and if the average value of x^k on $[0, k]$ is k, what is k? Check your result with a CAS if you have one available.

53. Show that if $F'(x) = G'(x)$ on $[a, b]$, then

$$F(b) - F(a) = G(b) - G(a).$$