6.3. Average Value of a Function

The average value $\overline{f}$ of $f(x)$ on $x \in [a, b]$ is

$$\boxed{\overline{f} = \frac{1}{b-a} \int_a^b f(x)\,dx}$$

or $\int_a^b f(x)\,dx = \overline{f}(b-a)$ is the area under the curve.

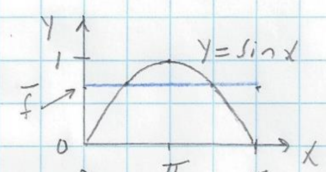
SIDE NOTE: Proof of the average rate of change.

$$\overline{f'} = \frac{1}{b-a} \int_a^b f'(x)\,dx = \frac{f(b)-f(a)}{b-a}$$

Example #1. Calculate the average value of $\sin x$ over $x \in [0, \pi]$.

SOLUTION: $f(x) = \sin x$,

$$\overline{f} = \frac{1}{\pi - 0} \int_0^\pi \sin x\,dx = \frac{1}{\pi} \Big[-\cos x\Big]_0^\pi = \frac{1}{\pi}\Big[-\cos\pi - -\cos 0\Big] = \frac{1}{\pi}\Big[--1+1\Big] = \frac{2}{\pi} \simeq 0.6366$$



Mean Value Theorem for Definite Integrals (MVTDI)

If $f(x)$ is continuous on $x \in [a, b]$, then there exists at least one $c \in [a, b]$ where

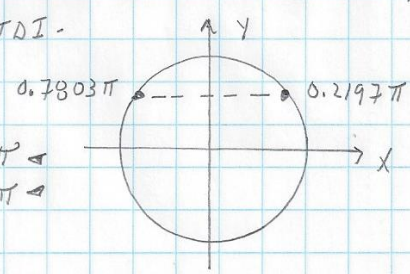
$$f(c) = \overline{f} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

Example #2. For $f(x) = \sin x$ on $x \in [0, \pi]$, find the values of $c \in [0, \pi]$ which are guaranteed to exist by the MVTDI.

SOLUTION:

$f(c) = \overline{f}$, $\sin c = \frac{2}{\pi}$ $\quad c = \sin^{-1}\left(\frac{2}{\pi}\right) = 0.2197\pi$

$= 0.7803\pi$



6.3. Average Value of a Function

CLASS WORK

For $f(x) = 4x - x^2$ on $x \in [0,4]$

(1) Calculate $\bar{f}$.

(2) Find the values of $c \in [0,4]$ which are guaranteed to exist by the MVTDI

(3) Graph $y = f(x)$ on $x \in [0,4]$ and indicate the value of $\bar{f}$ on the graph.

SOLUTIONS

(1) $\bar{f} = \frac{1}{4-0} \int_0^4 (4x - x^2)\,dx = \frac{1}{4}\left[2x^2 - \frac{1}{3}x^3\right]_0^4 = \frac{1}{4}\left[10\frac{2}{3} - 0\right] = 2\frac{2}{3} = \frac{8}{3}$

(2) $f(c) = \bar{f}$, $4c - c^2 = \frac{8}{3}$, $c^2 - 4c + \frac{8}{3} = 0$, $c = \frac{4 \pm \sqrt{4^2 - 4(1)\left(\frac{8}{3}\right)}}{2(1)} = \frac{4 \pm \sqrt{\frac{16}{3}}}{2} =$

$= \frac{4 \pm \frac{4}{\sqrt{3}}}{2} = 2 \pm \frac{2}{\sqrt{3}}$ $\quad c = 0.8453$ $\quad c = 3.1547$

(3)