6.4. Integrals of Piecewise-Defined Functions

For each example, given $f'(x)$:

(a) Graph $y = f'(x)$.

(b) Calculate $f(x) = f(0) + \int_0^x f'(t)\,dt$, using the condition $f(0) = 0$, on $x \in [0,4]$.

(c) Graph $y = f(x)$.

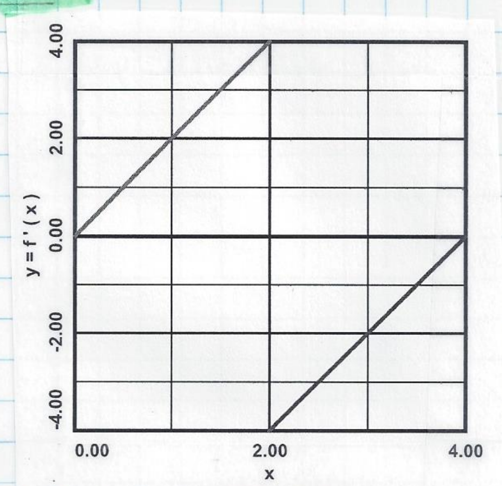
(d) Is $y = f(x)$ differentiable at $x = 2$?

Example #1.

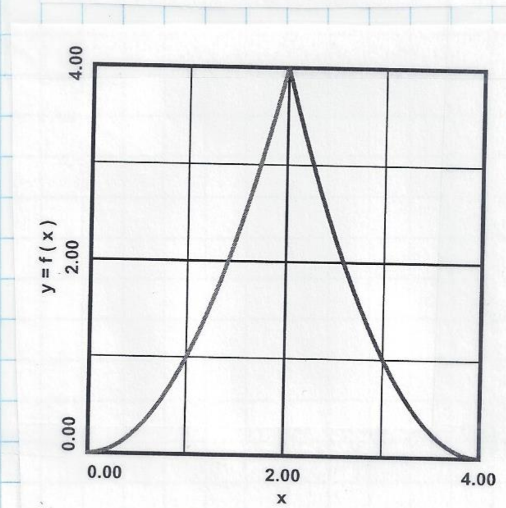
$$f'(x) = \begin{cases} 2x, & 0 \le x < 2 \\ 2x - 8, & 2 < x \le 4 \end{cases}$$

SOLUTION:

(a)



(c)



(b) $0 \le x \le 2$:

$$f(x) = \cancel{f(0)} + \int_0^x f'(t)\,dt =$$

$$= \int_0^x (2t)\,dt = [t^2]_0^x = x^2$$

$f(2) = 4$

$2 < x \le 4$:

$$f(x) = f(2) + \int_2^x f'(t)\,dt =$$

$$= 4 + \int_2^x (2t - 8)\,dt =$$

$$= 4 + [t^2 - 8t]_2^x = 4 + x^2 - 8x - -12 =$$

$$= x^2 - 8x + 16$$

$$f(x) = \begin{cases} x^2, & 0 \le x \le 2 \\ x^2 - 8x + 16, & 2 < x \le 4 \end{cases}$$

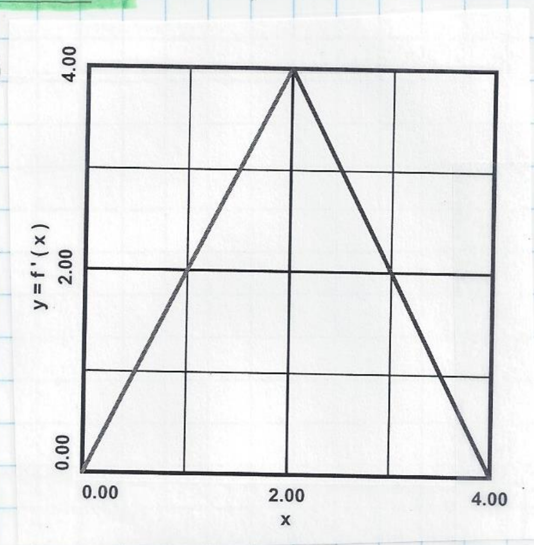
(d) While $f(x)$ is continuous at $x = 2$, it is not differentiable (due to the kink).

Example #2.

$$f'(x) = \begin{cases} 2x & , \ 0 \le x \le 2 \\ -2x + 8 & , \ 2 < x \le 4 \end{cases}$$

SOLUTION:

(a)



(b) $0 \le x \le 2$:

$f(x) = x^2 \qquad f(2) = 4$

$2 \le x \le 4$:

$$f(x) = f(2) + \int_2^x f'(t)\,dt =$$

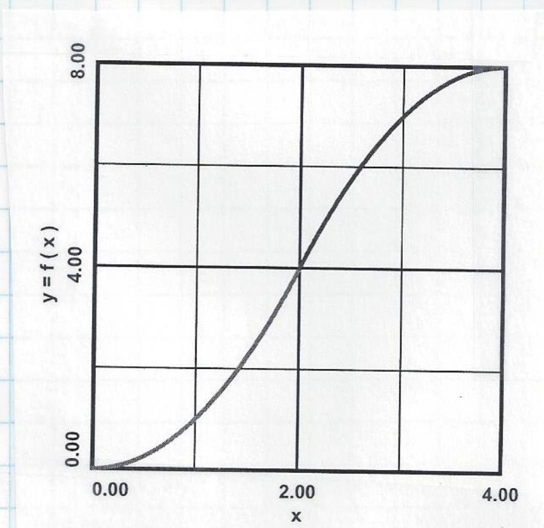
$$= 4 + \int_2^x (-2t + 8)\,dt =$$

$$= 4 + \left[-t^2 + 8t\right]_2^x =$$

$$= 4 - x^2 + 8x - 12 = -x^2 + 8x - 8$$

$$f(x) = \begin{cases} x^2 & , \ 0 \le x \le 2 \\ -x^2 + 8x - 8 & , \ 2 < x \le 4. \end{cases}$$

(c)



(d) $y = f(x)$ is smooth, i.e., differentiable at $x = 2$.

In fact, $f'(2^-) = 4$ and $f'(2^+) = 4$.

Note:

(1) The integral of a jump gives a kink.

(2) The integral of a kink is smooth.