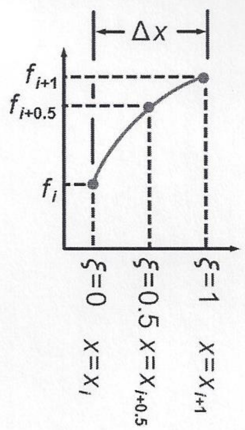
As before, in order to estimate the value of the integral

$$I = \int_A^B f(x)\,\mathrm{d}x\,,$$

we break up the interval $x \in [A\,,B]$ into N equal subdivisions of size Δx, where

$$\Delta x = \frac{B-A}{N}\,.$$

The figure below shows the ith subdivision ($i = 1,2,\cdots,N$) on $x \in [x_i\,,x_{i+1}]$, where



$$x_i = A + (i-1)\Delta x$$

$$x_{i+0.5} = x_i + \frac{\Delta x}{2} \qquad x_{i+1} = x_i + \Delta x\,.$$

Note that the figure also has the variable $\xi \in [0\,,1]$, where the relation between ξ and x is given by

$$x = (\Delta x)\xi + x_i \qquad \frac{\mathrm{d}x}{\mathrm{d}\xi} = \Delta x\,.$$

The idea behind Simpson's Rule is to approximate $f(x)$ as a quadratic function on the subinterval, *viz.*,

$$f(\xi) = (2f_i - 4f_{i+0.5} + 2f_{i+1})\xi^2 + \\ +(-3f_i + 4f_{i+0.5} - f_{i+1})\xi + f_i\,.$$

1) Show that $f(\xi)$ does in fact correspond to the figure, *i.e.*, show

a) $f(\xi = 0) = f_i$ $\quad f(0) = 0 + 0 + f_i = f_i$

b) $f(\xi = 0.5) = f_{i+0.5}$

c) $f(\xi = 1) = f_{i+1}$

$$(b)\ f\left(\tfrac{1}{2}\right) = \tfrac{1}{4}(2f_i - 4f_{i+0.5} + 2f_{i+1}) + \tfrac{1}{2}(-3f_i + 4f_{i+0.5} - f_{i+1}) + f_i =$$

$$= \tfrac{1}{4}(\cancel{2f_i} - 4f_{i+0.5} + \cancel{2f_{i+1}} - \cancel{6f_i} + 8f_{i+0.5} - \cancel{2f_{i+1}} + \cancel{4f_i})$$

$$= \tfrac{1}{4}(4f_{i+0.5}) = f_{i+0.5}$$

$$(c)\ f(1) = \cancel{2f_i} - \cancel{4f_{i+0.5}} + 2f_{i+1} - \cancel{3f_i} + \cancel{4f_{i+0.5}} - f_{i+1} + \cancel{f_i} =$$

$$= f_{i+1}$$

Now, the integral over the ith subdivision is

$$I_i = \int_{x_i}^{x_{i+1}} f(x)\,\mathrm{d}x = \int_0^1 f(\xi)\,\frac{\mathrm{d}x}{\mathrm{d}\xi}\,\mathrm{d}\xi = \\ = \Delta x \int_0^1 f(\xi)\,\mathrm{d}\xi\,.$$

2) Perform the integration to show that

$$I_i = \frac{\Delta x}{6}(f_i + 4f_{i+0.5} + f_{i+1})$$

so that

$$I = \sum_{i=1}^{N} I_i\,.$$

$$I_i = \Delta x \left[\frac{\xi^3}{3}(2f_i - 4f_{i+0.5} + 2f_{i+1}) + \frac{\xi^2}{2}(-3f_i + 4f_{i+0.5} - f_{i+1}) + \xi f_i \right]_0^1 =$$

$$= \frac{\Delta x}{6}\left[(4f_i - 8f_{i+0.5} + 4f_{i+1})\xi^3 + (-9f_i + 12f_{i+0.5} - 3f_{i+1})\xi^2 + (6f_i)\xi \right]_0^1 =$$

$$= \frac{\Delta x}{6}(4f_i - 8f_{i+0.5} + 4f_{i+1} - 9f_i + 12f_{i+0.5} - 3f_{i+1} + 6f_i) =$$

$$= \frac{\Delta x}{6}(f_i + 4f_{i+0.5} + f_{i+1})$$