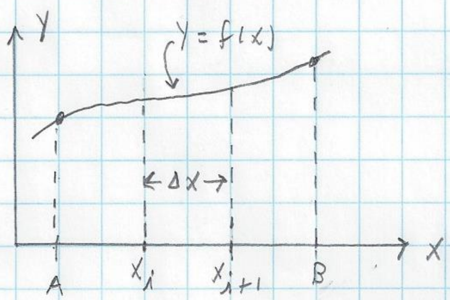
Polynomial-Based Integration Rules



$$I = \int_A^B f(x)\,dx \approx \sum_{i=1}^{N} I_i$$

Break the interval $x \in [A, B]$ into N equal subdivisions of size

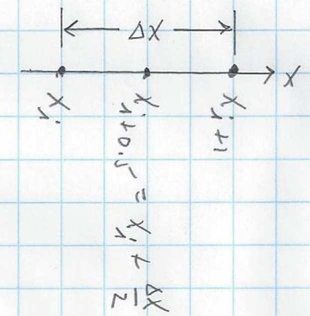
$$\Delta x = \frac{B-A}{N},$$ where

$$I_i = \int_{x_i}^{x_{i+1}} f(x)\,dx \equiv \text{integral over the } i^{th} \text{ subdivision}$$

$x_i = A + (i-1)\Delta x \quad (i = 1, 2, \cdots, N) \equiv$ left end-point of subdivision i

$x_{i+1} = x_i + \Delta x \qquad \equiv$ right end-point of subdivision i

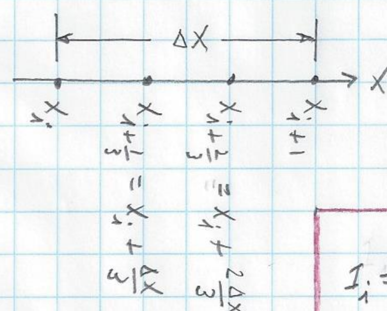
Quadratic (Simpson's) Rule



Fit the function values $f(x_i)$, $f(x_{i+0.5})$ and $f(x_{i+1})$ to a quadratic $\Rightarrow$

$$\boxed{I_i = \frac{\Delta x}{6}\left[f(x_i) + 4f(x_{i+0.5}) + f(x_{i+1})\right]}$$

Cubic Rule



Fit the function values $f(x_i)$, $f(x_{i+\frac{1}{3}})$, $f(x_{i+\frac{2}{3}})$ and $f(x_i)$ to a cubic polynomial $\Rightarrow$

$$\boxed{I_i = \frac{\Delta x}{8}\left[f(x_i) + 3f(x_{i+\frac{1}{3}}) + 3f(x_{i+\frac{2}{3}}) + f(x_{i+1})\right]}$$

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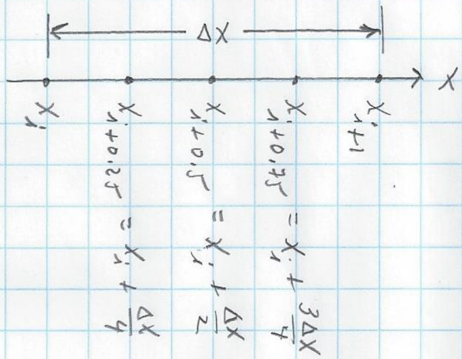
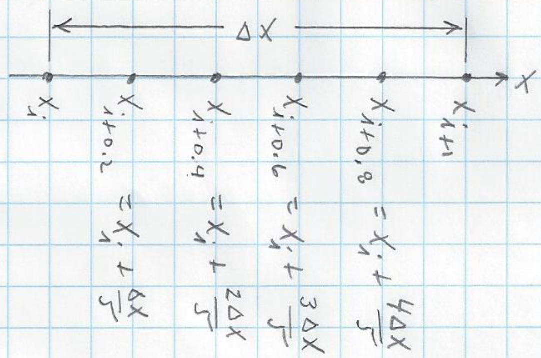
Quartic Rule



Fit the function values $f(x_i)$, $f(x_{i+0.25})$, $f(x_{i+0.5})$, $f(x_{i+0.75})$ and $f(x_{i+1})$ to a quartic polynomial $\Rightarrow$

$$\boxed{I_i = \frac{\Delta x}{90}\left[7f(x_i) + 32f(x_{i+0.25}) + 12f(x_{i+0.5}) + 32f(x_{i+0.75}) + 7f(x_{i+1})\right]}$$

Quintic Rule



Fit the function values $f(x_i)$, $f(x_{i+0.2})$, $f(x_{i+0.4})$, $f(x_{i+0.6})$, $f(x_{i+0.8})$ and $f(x_{i+1})$ to a quintic polynomial $\Rightarrow$

$$\boxed{I_i = \frac{\Delta x}{288}\left[19f(x_i) + 75f(x_{i+0.2}) + 50f(x_{i+0.4}) + 50f(x_{i+0.6}) + 75f(x_{i+0.8}) + 19f(x_{i+1})\right]}$$

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Example: Use the above rules to estimate

$$I = \int_0^1 \frac{\pi}{4} \sec^2\left(\frac{\pi x}{4}\right) dx = \left[\tan\left(\frac{\pi x}{4}\right)\right]_0^1 = 1,$$

i.e., $f(x) = \frac{\pi}{4} \sec^2\left(\frac{\pi x}{4}\right)$, with 9 subdivisions ($N=9$, $A=0$, $B=1$).

SOLUTION:

Rule	I
Quadratic	1.000 001 593
Cubic	1.000 000 709
Quartic	1.000 000 001
Quintic	1.000 000 000