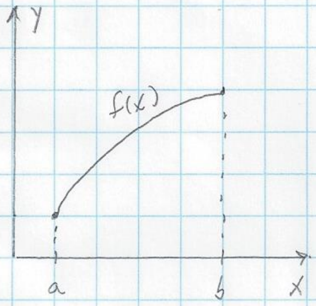
6.2. The Definite Integral



Using RAM to estimate the area under $y=f(x)$ on $x\in[a,b]$ we had

$$\text{area} \approx \sum_{i=1}^{N} f(x_i+\theta)\Delta x \qquad \Delta x = \frac{b-a}{N}$$

$$\theta \equiv \text{offset} \qquad 0 \le \theta \le \Delta x$$

Letting $N\to\infty$ gives the exact area...

$$\text{area} = \lim_{\substack{N\to\infty \\ \Delta x \to dx}} \sum_{i=1}^{N} f(x_i+\theta)\Delta x \equiv \int_a^b f(x)\,dx$$

$dx \equiv$ differential of x
$\equiv$ an infinitesimal length along the x-axis.

upper limit

the integrand

Integral Sign. It is a stylized "S", for sum.

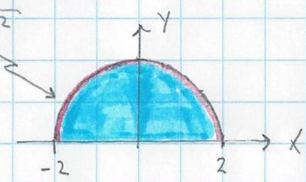
$$\int_a^b f(x)\,dx$$ is called the *definite integral* from a to b of $f(x)$.

lower limit

Example #1. Calculate $\displaystyle\int_{-2}^{2} \sqrt{4-x^2}\,dx$.

SOLUTION: $x^2+y^2=2^2=4$ is a circle of radius 2 centered at the origin. The upper half of the circle is...

$y^2 = 4-x^2$ or $y=\sqrt{4-x^2}$

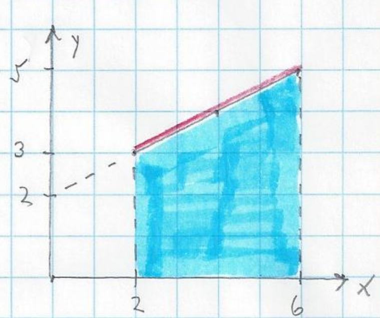


So, the integral is one-half the area of a circle $\Rightarrow$

$$\int_{-2}^{2} \sqrt{4-x^2}\,dx = \tfrac{1}{2}\pi r^2 = \tfrac{1}{2}\pi 2^2 = 2\pi$$

Example #2. Calculate $\int_2^6 \left(\frac{1}{2}x+2\right)dx$.

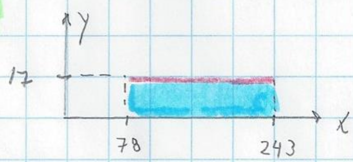
SOLUTION: The area is a trapezoid...



$$\int_2^6 \left(\frac{1}{2}x+2\right)dx = \frac{1}{2}(b_1+b_2)h = \frac{1}{2}(3+5)(4) = 16$$

Example #3. Calculate $\int_{78}^{243} 17\,dx$.

SOLUTION:



It is the area of a rectangle...

$$\int_{78}^{243} 17\,dx = bh = (243-78)(17) =$$
$$= (165)(17) = 2805$$

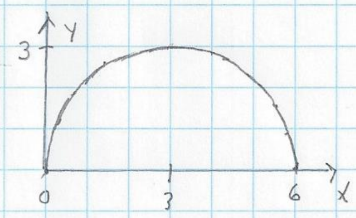
CLASS WORK Calculate the integrals.

(1) $\int_0^6 \sqrt{9-(x-3)^2}\,dx$ (2) $\int_1^3 (2x+1)\,dx$ (3) $\int_{32}^{98} 43\,dx$

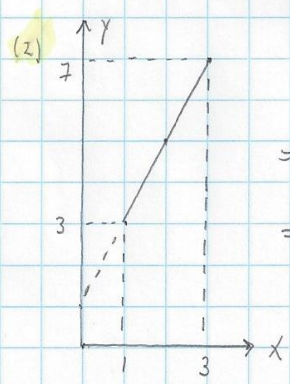
SOLUTIONS

(1) $y=\sqrt{9-(x-3)^2}$, $y^2 = 9-(x-3)^2$, $(x-3)^2+y^2=3^2$

$\frac{1}{2}\pi r^2 = \frac{1}{2}\pi 3^2 = \frac{9}{2}\pi$



(2)



$\frac{1}{2}(b_1+b_2)h =$

$= \frac{1}{2}(3+7)(2) =$

$= 10$

(3) $bh = (98-32)(43) =$
$= (66)(43) = 2838$