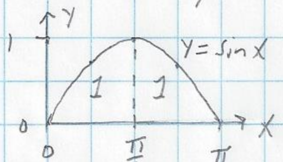
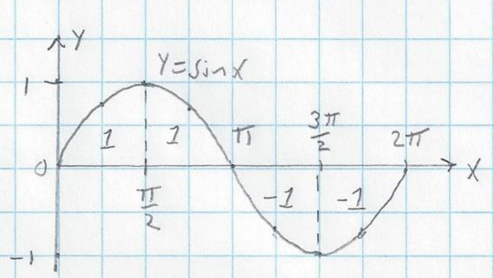
6.2 & 6.3: Negative Areas & Properties of Integrals

Two lessons previous we found, by using MRAM, that $\int_0^{\pi/2} \sin x\,dx = 1$.



$$\Rightarrow \int_0^{\pi} \sin x\,dx = 2$$



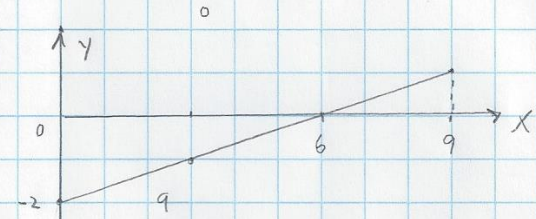
Areas below the x-axis are negative

so, for example,

$$\int_0^{\frac{3\pi}{2}} \sin x\,dx = 1+1-1 = \underline{1} \qquad \int_0^{2\pi} \sin x\,dx = 1+1-1-1 = 0$$

Example #1. Calculate $\int_0^9 \left(\frac{1}{3}x - 2\right)dx$.

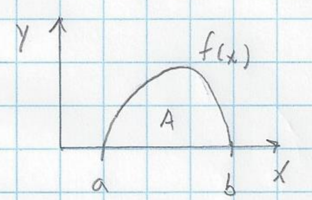
SOLUTION:



We have two triangles...

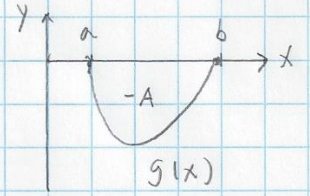
$$\int_0^9 \left(\tfrac{1}{3}x-2\right)dx = -\tfrac{1}{2}b_1h_1 + \tfrac{1}{2}b_2h_2 = -\tfrac{1}{2}(6)(2) + \tfrac{1}{2}(3)(1) =$$
$$= -6 + \frac{3}{2} = -4\tfrac{1}{2}$$

dx can be positive or negative $(A > 0)$



$\int_a^b f(x)\,dx = A$ because $f(x) > 0$ and $dx > 0$

$\int_b^a f(x)\,dx = -A$ because $f(x) > 0$ and $dx < 0$



$\int_a^b g(x)\,dx = -A$ because $g(x) < 0$ and $dx > 0$

$\int_b^a g(x)\,dx = A$ because $g(x) < 0$ and $dx < 0$

So, we have the property

$$\boxed{(1)\quad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx}$$

These properties are really just common sense . . .

$$(2)\quad \int_a^a f(x)\,dx = 0 \quad (\text{because } dx = 0)$$

$$(3)\quad \int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

These properties follow from the RAM definition

$$\int_a^b f(x)\,dx = \lim_{\substack{N\to\infty \\ \Delta x \to dx}} \sum_{i=1}^{N} f(x_i + \theta)\,\Delta x$$

$$(4)\quad \int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx \quad (k \equiv \text{constant})$$

$$(5)\quad \int_a^b \left[f(x) \pm g(x)\right] dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

Example #2. Given that

$$\int_0^3 f(x)\,dx = 4, \quad \int_0^6 f(x)\,dx = 3 \quad \text{and} \quad \int_3^6 g(x)\,dx = 8,$$

calculate

(a) $\int_3^6 f(x)\,dx$ (b) $\int_6^3 f(x)\,dx$ (c) $\int_3^6 [4f(x) - g(x)]\,dx$

SOLUTION!

(a) $\int_0^6 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^6 f(x)\,dx$, $3 = 4 + \int_3^6 f(x)\,dx$, $\int_3^6 f(x)\,dx = -1$ ◄

(b) $\int_6^3 f(x)\,dx = -\int_3^6 f(x)\,dx = --1 = 1$ ◄

(c) $\int_3^6 [4f(x) - g(x)]\,dx = \int_3^6 4f(x)\,dx - \int_3^6 g(x)\,dx = 4\int_3^6 f(x)\,dx - \int_3^6 g(x)\,dx =$

$$= 4(-1) - 8 = -12 \text{ ◄}$$

CLASS WORK

(1) Compare the graphs of $y = \sin x$ and $y = \cos x$ to calculate

(a) $\int_0^{\pi/2} \cos x\,dx$ (b) $\int_0^{\pi} \cos x\,dx$ (c) $\int_0^{3\pi/2} \cos x\,dx$ (d) $\int_0^{2\pi} \cos x\,dx$

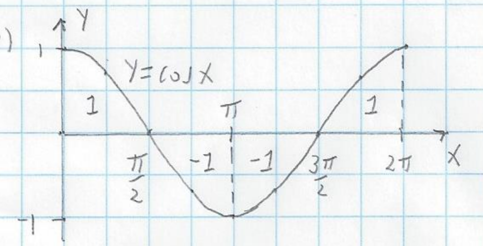
(2) Calculate $\int_0^8 \left(-\frac{1}{2}x + 3\right)dx$.

(3) Given that $\int_2^7 f(x)\,dx = 18$, $\int_5^7 f(x)\,dx = -2$ and $\int_2^5 g(x)\,dx = 13$, calculate

(a) $\int_2^5 f(x)\,dx$ (b) $\int_5^2 3f(x)\,dx$ (c) $\int_2^5 [5f(x) + 7g(x)]\,dx$

SOLUTIONS

(1)



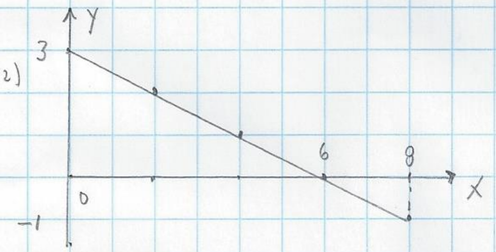
a) $\int_0^{\pi/2} \cos x \, dx = 1$

b) $\int_0^{\pi} \cos x \, dx = 1 - 1 = 0$

c) $\int_0^{3\pi/2} \cos x \, dx = 1 - 1 - 1 = -1$

d) $\int_0^{2\pi} \cos x \, dx = 1 - 1 - 1 + 1 = 0$

(2)



$$\int_0^8 \left(-\tfrac{1}{2}x + 3\right) dx = \tfrac{1}{2} b_1 h_1 - \tfrac{1}{2} b_2 h_2 =$$

$$= \tfrac{1}{2}(6)(3) - \tfrac{1}{2}(2)(1) = 9 - 1 = 8$$

(3) (a) $\int_2^7 f(x)\,dx = \int_2^5 f(x)\,dx + \int_5^7 f(x)\,dx$, $18 = \int_2^5 f(x)\,dx + -2$, $\int_2^5 f(x)\,dx = 20$

(b) $\int_5^2 3f(x)\,dx = -\int_2^5 3f(x)\,dx = -3\int_2^5 f(x)\,dx = -3(20) = -60$

(c) $\int_2^5 [5f(x) + 7g(x)]\,dx = \int_2^5 5f(x)\,dx + \int_2^5 7g(x)\,dx = 5\int_2^5 f(x)\,dx + 7\int_2^5 g(x)\,dx =$

$= 5(20) + 7(13) = 191$