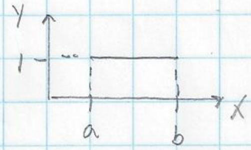
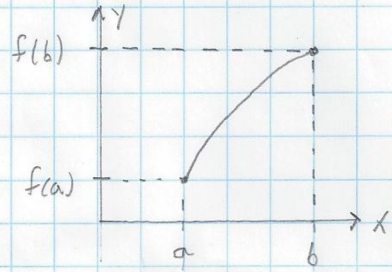
6.4. The Fundamental Theorem of Calculus (FTC)



$$\int_a^b 1 \cdot dx = \int_a^b dx = 1 \cdot (b-a) = b-a$$

We can do the same kind of thing along the y-direction



$$\int_{\substack{y=f(a)\\x=a}}^{\substack{y=f(b)\\x=b}} df = f(b) - f(a) \qquad \frac{df}{dx} = f'(x) \qquad df = f'(x)\,dx \Rightarrow$$

$$\int_{\substack{y=f(a)\\x=a}}^{\substack{y=f(b)\\x=b}} f'(x)\,dx = f(b) - f(a) \quad \text{or} \quad \boxed{\int_a^b f'(x)\,dx = f(b) - f(a)}$$

FTC.1

Change x to t $\Rightarrow$ $f(b) = f(a) + \int_a^b f'(t)\,dt$,

t is called the integration variable (or dummy variable)

and now change b to x

$$\boxed{f(x) = f(a) + \int_a^x f'(t)\,dt}$$

FTC.2

$f(x)$ is called the antiderivative of $f'(x)$

Given $f'(x)$, the process of finding $f(x)$ is called antidifferentiation.

Take the derivative of FTC.2 $\Rightarrow$ $f'(x) = \frac{d}{dx}\int_a^x f'(t)\,dt$ or

$$\boxed{f(x) = \frac{d}{dx}\int_a^x f(t)\,dt}$$

FTC.3

We can that differentiation and integration are inverse operations, i.e., integration undoes differentiation (FTC.2), and differentiation undoes integration (FTC.3).

6.4. The Fundamental Theorem of Calculus (FTC)

Power Rule

$$\boxed{\text{If } f'(x) = x^p, \text{ then } f(x) = \frac{x^{p+1}}{p+1}}$$

$$f'(x) = \frac{1}{\cancel{p+1}} \cdot \cancel{(p+1)}\, x^{p+1-1} = x^p$$

Short-hand Notation

$$f(b) - f(a) = \Big[f(x)\Big]_a^b$$

so FTC.1 can be written as $\displaystyle\int_a^b f'(x)\,dx = \Big[f(x)\Big]_a^b$.

Example #1. Calculate $\displaystyle\int_0^2 (12x - 3x^2)\,dx$

SOLUTION: $f'(x) = 12x - 3x^2$, $f(x) = 12\frac{x^2}{2} - 3\frac{x^3}{3} = 6x^2 - x^3$

$$\int_0^2 (12x - 3x^2)\,dx = \Big[6x^2 - x^3\Big]_0^2 = 16 - 0 = 16$$

Example #2. Calculate $\displaystyle\int_0^{\frac{\pi}{2}} \sin x\,dx$.

SOLUTION: $f'(x) = \sin x$ $f(x) = -\cos x$

$$\int_0^{\pi/2} \sin x\,dx = \Big[-\cos x\Big]_0^{\pi/2} = -\cos\frac{\pi}{2} - -\cos 0 = 0 + 1 = 1$$

CLASS WORK

Calculate the definite integrals

(1) $\displaystyle\int_0^4 (4x - x^2)\,dx$ (2) $\displaystyle\int_0^{\pi/4} \sec^2 x\,dx$

SOLUTIONS

(1) $f'(x) = 4x - x^2$, $f(x) = 4\frac{x^2}{2} - \frac{x^3}{3} = 2x^2 - \frac{1}{3}x^3$

$$\int_0^4 (4x - x^2)\,dx = \Big[2x^2 - \frac{1}{3}x^3\Big]_0^4 = 10\tfrac{2}{3} - 0 = 10\tfrac{2}{3}$$

6.4. The Fundamental Theorem of Calculus

(2) $f'(x) = \sec^2 x$, $f(x) = \tan x$

$$\int_0^{\pi/4} \sec^2 x \, dx = \Big[\tan x\Big]_0^{\pi/4} = \tan\frac{\pi}{4} - \tan 0 = 1 - 0 = 1$$