Section 7.4 Exercises

In Exercises 1–10, use separation of variables to solve the initial value problem. Indicate the domain over which the solution is valid.

1. $\dfrac{dy}{dx} = \dfrac{x}{y}$ and $y = 2$ when $x = 1$

2. $\dfrac{dy}{dx} = -\dfrac{x}{y}$ and $y = 3$ when $x = 4$

3. $\dfrac{dy}{dx} = \dfrac{y}{x}$ and $y = 2$ when $x = 2$

4. $\dfrac{dy}{dx} = 2xy$ and $y = 3$ when $x = 0$

5. $\dfrac{dy}{dx} = (y + 5)(x + 2)$ and $y = 1$ when $x = 0$

6. $\dfrac{dy}{dx} = \cos^2 y$ and $y = 0$ when $x = 0$

7. $\dfrac{dy}{dx} = (\cos x)e^{y+\sin x}$ and $y = 0$ when $x = 0$

8. $\dfrac{dy}{dx} = e^{x-y}$ and $y = 2$ when $x = 0$

9. $\dfrac{dy}{dx} = -2xy^2$ and $y = 0.25$ when $x = 1$

10. $\dfrac{dy}{dx} = \dfrac{4\sqrt{y}\ln x}{x}$ and $y = 1$ when $x = e$

In Exercises 11–14, find the solution of the differential equation $dy/dt = ky$, k a constant, that satisfies the given conditions.

11. $k = 1.5,\ y(0) = 100$ **12.** $k = -0.5,\ y(0) = 200$

13. $y(0) = 50,\ y(5) = 100$ **14.** $y(0) = 60,\ y(10) = 30$

In Exercises 15–18, complete the table for an investment if interest is compounded continuously.

	Initial Deposit ($)	Annual Rate (%)	Doubling Time (yr)	Amount in 30 yr ($)
15.	1000	8.6		
16.	2000		15	
17.		5.25		2898.44
18.	1200			10,405.37

In Exercises 19 and 20, find the amount of time required for a $2000 investment to double if the annual interest rate r is compounded **(a)** annually, **(b)** monthly, **(c)** quarterly, and **(d)** continuously.

19. $r = 4.75\%$ **20.** $r = 8.25\%$

21. ***Half-Life*** The radioactive decay of Sm-151 (an isotope of samarium) can be modeled by the differential equation $dy/dt = -0.0077y$, where t is measured in years. Find the half-life of Sm-151.

22. ***Half-Life*** An isotope of neptunium (Np-240) has a half-life of 65 minutes. If the decay of Np-240 is modeled by the differential equation $dy/dt = -ky$, where t is measured in minutes, what is the decay constant k?

23. ***Growth of Cholera Bacteria*** Suppose that the cholera bacteria in a colony grow unchecked according to the Law of Exponential Change. The colony starts with 1 bacterium and doubles in number every half hour.

(a) How many bacteria will the colony contain at the end of 24 h?

(b) **Writing to Learn** Use part (a) to explain why a person who feels well in the morning may be dangerously ill by evening even though, in an infected person, many bacteria are destroyed.

24. ***Bacterial Growth*** A colony of bacteria is grown under ideal conditions in a laboratory so that the population increases exponentially with time. At the end of 3 h there are 10,000 bacteria. At the end of 5 h there are 40,000 bacteria. How many bacteria were present initially?

25. ***Radon-222*** The decay equation for radon-222 gas is known to be $y = y_0e^{-0.18t}$, with t in days. About how long will it take the amount of radon in a sealed sample of air to decay to 90% of its original value?

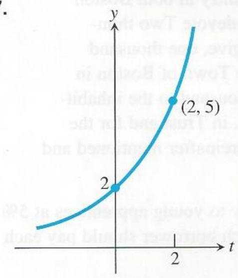
26. ***Polonium-210*** The number of radioactive atoms remaining after t days in a sample of polonium-210 that starts with y_0 radioactive atoms is $y = y_0e^{-0.005t}$.

(a) Find the element's half-life.

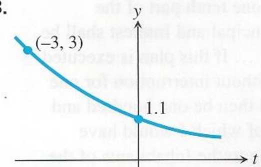
(b) Your sample will not be useful to you after 95% of the radioactive nuclei present on the day the sample arrives have disintegrated. For about how many days after the sample arrives will you be able to use the polonium?

In Exercises 27 and 28, find the exponential function $y = y_0 e^{kt}$ whose graph passes through the two points.

27.



28.



29. *Mean Life of Radioactive Nuclei* Physicists using the radioactive decay equation $y = y_0 e^{-kt}$ call the number $1/k$ the *mean life* of a radioactive nucleus. The mean life of a radon-222 nucleus is about $1/0.18 \approx 5.6$ days. The mean life of a carbon-14 nucleus is more than 8000 years. Show that 95% of the radioactive nuclei originally present in any sample will disintegrate within three mean lifetimes, that is, by time $t = 3/k$. Thus, the mean life of a nucleus gives a quick way to estimate how long the radioactivity of a sample will last.

30. *Finding the Original Temperature of a Beam* An aluminum beam was brought from the outside cold into a machine shop where the temperature was held at 65°F. After 10 min, the beam warmed to 35°F and after another 10 min its temperature was 50°F. Use Newton's Law of Cooling to estimate the beam's initial temperature.

31. *Cooling Soup* Suppose that a cup of soup cooled from 90°C to 60°C in 10 min in a room whose temperature was 20°C. Use Newton's Law of Cooling to answer the following questions.

(a) How much longer would it take the soup to cool to 35°C?

(b) Instead of being left to stand in the room, the cup of 90°C soup is put into a freezer whose temperature is −15°C. How long will it take the soup to cool from 90°C to 35°C?

32. *Cooling Silver* The temperature of an ingot of silver is 60°C above room temperature right now. Twenty minutes ago, it was 70°C above room temperature. How far above room temperature will the silver be

(a) 15 minutes from now?

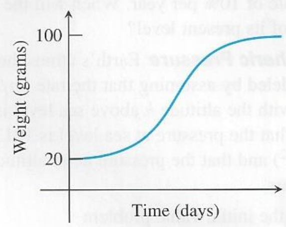
(b) 2 hours from now?

(c) When will the silver be 10°C above room temperature?

33. *Avian Development* The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, t days after it is first weighed, then $dB/dt = 0.2(100 - B)$.

(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

(b) Find $\dfrac{d^2B}{dt^2}$ in terms of t. Use $\dfrac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph:



(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.

34. *Solid Waste Accumulation* A landfill at time $t = 0$ years contains 1200 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential equation $dW/dt = 0.04(W - 300)$ for the years $0 \le t \le 20$, where W is measured in tons.

(a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste in the landfill 3 months later.

(b) Find $\dfrac{d^2W}{dt^2}$ in terms of W. Use $\dfrac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = 1/4$.

(c) Find the particular solution $W = W(t)$ to the differential equation $dW/dt = 0.04(W - 300)$ with initial condition $W(0) = 1200$.

35. *Dating Crater Lake* The charcoal from a tree killed in the volcanic eruption that formed Crater Lake in Oregon contained 44.5% of the carbon-14 found in living matter. About how old is Crater Lake?

36. *Carbon-14 Dating Measurement Sensitivity* To see the effect of a relatively small error in the estimate of the amount of carbon-14 in a sample being dated, answer the following questions about this hypothetical situation.

(a) A fossilized bone found in central Illinois in the year 2000 C.E. contains 17% of its original carbon-14 content. Estimate the year the animal died.

(b) Repeat part (a) assuming 18% instead of 17%.

(c) Repeat part (a) assuming 16% instead of 17%.

37. What is the half-life of a substance that decays to $1/3$ of its original radioactive amount in 5 years?

38. A savings account earning compound interest triples in value in 10 years. How long will it take for the original investment to quadruple?

39. *The Inversion of Sugar* The processing of raw sugar has an "inversion" step that changes the sugar's molecular structure. Once the process has begun, the rate of change of the amount of raw sugar is proportional to the amount of raw sugar remaining. If 1000 kg of raw sugar reduces to 800 kg of raw sugar during the first 10 h, how much raw sugar will remain after another 14 h?

40. *Oil Depletion* Suppose the amount of oil pumped from one of the canyon wells in Whittier, California, decreases at the continuous rate of 10% per year. When will the well's output fall to one-fifth of its present level?

41. *Atmospheric Pressure* Earth's atmospheric pressure p is often modeled by assuming that the rate dp/dh at which p changes with the altitude h above sea level is proportional to p. Suppose that the pressure at sea level is 1013 millibars (about 14.7 lb/in^2) and that the pressure at an altitude of 20 km is 90 millibars.

(a) Solve the initial value problem

Differential equation: $\dfrac{dp}{dh} = kp,$

Initial condition: $p = p_0$ when $h = 0,$

to express p in terms of h. Determine the values of p_0 and k from the given altitude-pressure data.

(b) What is the atmospheric pressure at $h = 50$ km?

(c) At what altitude does the pressure equal 900 millibars?

42. *First-Order Chemical Reactions* In some chemical reactions the rate at which the amount of a substance changes with time is proportional to the amount present. For the change of δ-glucono lactone into gluconic acid, for example,

$$\frac{dy}{dt} = -0.6y$$

when y is measured in grams and t is measured in hours. If there are 100 grams of a δ-glucono lactone present when $t = 0$, how many grams will be left after the first hour?

43. *Discharging Capacitor Voltage* Suppose that electricity is draining from a capacitor at a rate proportional to the voltage V across its terminals and that, if t is measured in seconds,

$$\frac{dV}{dt} = -\frac{1}{40}V.$$

(a) Solve this differential equation for V, using V_0 to denote the value of V when $t = 0$.

(b) How long will it take the voltage to drop to 10% of its original value?

44. *John Napier's Answer* John Napier (1550–1617), the Scottish laird who invented logarithms, was the first person to answer the question, "What happens if you invest an amount of money at 100% yearly interest, compounded continuously?"

(a) Writing to Learn What does happen? Explain.

(b) How long does it take to triple your money?

(c) Writing to Learn How much can you earn in a year?

45. *Benjamin Franklin's Will* The Franklin Technical Institute of Boston owes its existence to a provision in a codicil to Benjamin Franklin's will. In part the codicil reads:

> I wish to be useful even after my Death, if possible, in forming and advancing other young men that may be serviceable to their Country in both Boston and Philadelphia. To this end I devote Two thousand Pounds Sterling, which I give, one thousand thereof to the Inhabitants of the Town of Boston in Massachusetts, and the other thousand to the inhabitants of the City of Philadelphia, in Trust and for the Uses, Interests and Purposes hereinafter mentioned and declared.

Franklin's plan was to lend money to young apprentices at 5% interest with the provision that each borrower should pay each year along

> . . . with the yearly Interest, one tenth part of the Principal, which sums of Principal and Interest shall be again let to fresh Borrowers. ... If this plan is executed and succeeds as projected without interruption for one hundred Years, the Sum will then be one hundred and thirty-one thousand Pounds of which I would have the Managers of the Donation to the Inhabitants of the Town of Boston, then lay out at their discretion one hundred thousand Pounds in Public Works. ... The remaining thirty-one thousand Pounds, I would have continued to be let out on Interest in the manner above directed for another hundred Years. ... At the end of this second term if no unfortunate accident has prevented the operation the sum will be Four Millions and Sixty-one Thousand Pounds.

It was not always possible to find as many borrowers as Franklin had planned, but the managers of the trust did the best they could. At the end of 100 years from the receipt of the Franklin gift, in January 1894, the fund had grown from 1000 pounds to almost 90,000 pounds. In 100 years the original capital had multiplied about 90 times instead of the 131 times Franklin had imagined.

(a) What annual rate of interest, compounded continuously for 100 years, would have multiplied Benjamin Franklin's original capital by 90?

(b) In Benjamin Franklin's estimate that the original 1000 pounds would grow to 131,000 in 100 years, he was using an annual rate of 5% and compounding once each year. What rate of interest per year when compounded continuously for 100 years would multiply the original amount by 131?

46. *Rules of 70 and 72* The rules state that it takes about $70/i$ or $72/i$ years for money to double at i percent, compounded continuously, using whichever of 70 or 72 is easier to divide by i.

(a) Show that it takes $t = (\ln 2)/r$ years for money to double if it is invested at annual interest rate r (in decimal form) compounded continuously.

(b) Graph the functions

$$y_1 = \frac{\ln 2}{r}, \quad y_2 = \frac{70}{100\,r}, \quad \text{and} \quad y_3 = \frac{72}{100r}$$

in the $[0, 0.1]$ by $[0, 100]$ viewing window.

(c) Writing to Learn Explain why these two rules of thumb for mental computation are reasonable.

(d) Use the rules to estimate how long it takes to double money at 5% compounded continuously.

(e) Invent a rule for estimating the number of years needed to triple your money.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

47. True or False If $dy/dx = ky$, then $y = e^{kx} + C$. Justify your answer.

48. True or False The general solution to $dy/dt = 2y$ can be written in the form $y = C(3^{kt})$ for some constants C and k. Justify your answer.

49. Multiple Choice A bank account earning continuously compounded interest doubles in value in 7.0 years. At the same interest rate, how long would it take the value of the account to triple?

(A) 4.4 years **(B)** 9.8 years **(C)** 10.5 years
(D) 11.1 years **(E)** 21.0 years

50. Multiple Choice A sample of Ce-143 (an isotope of cerium) loses 99% of its radioactive matter in 199 hours. What is the half-life of Ce-143?

(A) 4 hours **(B)** 6 hours **(C)** 30 hours
(D) 100.5 hours **(E)** 143 hours

51. Multiple Choice In which of the following models is dy/dt directly proportional to y?

I. $y = e^{kt} + C$
II. $y = Ce^{kt}$
III. $y = 2^{8kt}$

(A) I only **(B)** II only **(C)** I and II only
(D) II and III only **(E)** I, II, and III

52. Multiple Choice An apple pie comes out of the oven at 425°F and is placed on a counter in a 68°F room to cool. In 30 minutes it has cooled to 195°F. According to Newton's Law of Cooling, how many additional minutes must pass before it cools to 100°F?

(A) 12.4 **(B)** 15.4 **(C)** 25.0 **(D)** 35.0 **(E)** 40.0

Explorations

53. *Resistance Proportional to Velocity* It is reasonable to assume that the air resistance encountered by a moving object, such as a car coasting to a stop, is proportional to the object's velocity. The resisting force on an object of mass m moving with velocity v is thus $-kv$ for some positive constant k.

(a) Use the law *Force* = *Mass* × *Acceleration* to show that the velocity of an object slowed by air resistance (and no other forces) satisfies the differential equation

$$m\frac{dv}{dt} = -kv$$

(b) Solve the differential equation to show that $v = v_0 e^{-(k/m)t}$, where v_0 is the velocity of the object at time $t = 0$.

(c) If k is the same for two objects of different masses, which one will slow to half its starting velocity in the shortest time? Justify your answer.

54. *Coasting to a Stop* Assume that the resistance encountered by a moving object is proportional to the object's velocity so that its velocity is $v = v_0 e^{-(k/m)t}$.

(a) Integrate the velocity function with respect to t to obtain the distance function s. Assume that $s(0) = 0$ and show that

$$s(t) = \frac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right).$$

(b) Show that the total coasting distance traveled by the object as it coasts to a complete stop is $v_0 m/k$.

55. *Coasting to a Stop* Table 7.1 shows the distance s (meters) coasted in t seconds by Kelly Schmitzer on her in-line skates. Her initial velocity was $v_0 = 1$ m/sec, her mass was $m = 49.90$ kg (110 lb), and her total coasting distance was 1.32 m. Find a model for her position function using the form in Exercise 54(a), and superimpose the graph of the function on a scatter plot of the data. (*Note:* You should use your calculator to find the value of k in the formula, but do not use your calculator to fit a regression curve to the data.)

TABLE 7.1 Kelly Schmitzer Skating Data

t (sec)	0	0.1	0.3	0.5	0.7	0.9	1.1	1.3	1.5	1.7
s (m)	0	0.07	0.22	0.36	0.49	0.60	0.71	0.81	0.89	0.97

t (sec)	1.9	2.1	2.3	2.5	2.7	2.9	3.1	3.3	3.5	3.7
s (m)	1.05	1.11	1.17	1.22	1.25	1.28	1.30	1.31	1.32	1.32

Source: Valerie Sharritts, St. Francis de Sales H.S., Columbus, OH.

Extending the Ideas

56. *Continuously Compounded Interest*

(a) Use tables to give a numerical argument that

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e.$$

Support your argument graphically.

(b) For several different values of r, give numerical and graphical evidence that

$$\lim_{x\to\infty}\left(1 + \frac{r}{x}\right)^x = e^r.$$

(c) Writing to Learn Explain why compounding interest over smaller and smaller periods of time leads to the concept of interest compounded continuously.

57. *Skydiving* If a body of mass m falling from rest under the action of gravity encounters an air resistance proportional to the square of the velocity, then the body's velocity $v(t)$ is modeled by the initial value problem

Differential equation: $m\dfrac{dv}{dt} = mg - kv^2,$

Initial condition: $v(0) = 0,$

where t represents time in seconds, g is the acceleration due to gravity, and k is a constant that depends on the body's aerodynamic properties and the density of the air. (We assume that the fall is short enough so that variation in the air's density will not affect the outcome.)

(a) Show that the function

$$v(t) = \sqrt{\frac{mg}{k}}\,\frac{e^{at} - e^{-at}}{e^{at} + e^{-at}},$$

where $a = \sqrt{gk/m}$, is a solution of the initial value problem.

(b) Find the body's limiting velocity, $\lim_{t\to\infty} v(t)$.

(c) For a 160-lb skydiver $(mg = 160)$, and with time in seconds and distance in feet, a typical value for k is 0.005. What is the diver's limiting velocity in feet per second? in miles per hour?

Skydivers can vary their limiting velocities by changing the amount of body area opposing the fall. Their velocities can vary from 94 to 321 miles per hour.