Section 9.2 Exercises

In Exercises 1–4, estimate the limit graphically and then use l'Hospital's Rule to find the limit.

1. $\lim_{x \to 2} \frac{x-2}{x^2-4}$

2. $\lim_{x \to 0} \frac{\sin(5x)}{x}$

3. $\lim_{x \to 2} \frac{\sqrt{2+x}-2}{x-2}$

4. $\lim_{x \to 1} \frac{\sqrt[3]{x}-1}{x-1}$

In Exercises 5–8, apply the stronger form of l'Hospital's Rule to find the limit.

5. $\lim_{x \to 0} \frac{1-\cos x}{x^2}$

6. $\lim_{\theta \to \pi/2} \frac{1-\sin\theta}{1+\cos(2\theta)}$

7. $\lim_{t \to 0} \frac{\cos t - 1}{e^t - t - 1}$

8. $\lim_{x \to 2} \frac{x^2-4x+4}{x^3-12x+16}$

In Exercises 9–12, use l'Hospital's Rule to evaluate the one-sided limits. Support your answer graphically.

9. **(a)** $\lim_{x \to 0^-} \frac{\sin 4x}{\sin 2x}$ **(b)** $\lim_{x \to 0^+} \frac{\sin 4x}{\sin 2x}$

10. **(a)** $\lim_{x \to 0^-} \frac{\tan x}{x}$ **(b)** $\lim_{x \to 0^+} \frac{\tan x}{x}$

11. **(a)** $\lim_{x \to 0^-} \frac{\sin x}{x^3}$ **(b)** $\lim_{x \to 0^+} \frac{\sin x}{x^3}$

12. **(a)** $\lim_{x \to 0^-} \frac{\tan x}{x^2}$ **(b)** $\lim_{x \to 0^+} \frac{\tan x}{x^2}$

In Exercises 13–16, identify the indeterminate form and evaluate the limit using l'Hospital's Rule. Support your answer graphically.

13. $\lim_{x \to \pi} \frac{\csc x}{1+\cot x}$

14. $\lim_{x \to \pi/2} \frac{1+\sec x}{\tan x}$

15. $\lim_{x \to \infty} \frac{\ln(x+1)}{\log_2 x}$

16. $\lim_{x \to \infty} \frac{5x^2-3x}{7x^2+1}$

In Exercises 17–26, identify the indeterminate form and evaluate the limit using l'Hospital's Rule.

17. $\lim_{x \to 0^+} (x \ln x)$

18. $\lim_{x \to \infty} \left(x \tan \frac{1}{x}\right)$

19. $\lim_{x \to 0^+} (\csc x - \cot x + \cos x)$

20. $\lim_{x \to \infty} (\ln(2x) - \ln(x+1))$

21. $\lim_{x \to 0} (e^x + x)^{1/x}$

22. $\lim_{x \to 1} x^{1/(x-1)}$

23. $\lim_{x \to 1} (x^2 - 2x + 1)^{x-1}$

24. $\lim_{x \to 0^+} (\sin x)^x$

25. $\lim_{x \to 0^+} \left(1 + \frac{1}{x}\right)^x$

26. $\lim_{x \to \infty} (\ln x)^{1/x}$

In Exercises 27 and 28, **(a)** complete the table and estimate the limit. **(b)** Use l'Hospital's Rule to confirm your estimate.

27. $\lim_{x \to \infty} f(x)$, $f(x) = \frac{\ln x^5}{x}$

x	10	10^2	10^3	10^4	10^5
$f(x)$					

28. $\lim_{x\to 0^+} f(x),\ f(x) = \dfrac{x - \sin x}{x^3}$

x	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$f(x)$					

In Exercises 29–32, use tables to estimate the limit. Confirm your estimate using l'Hospital's Rule.

29. $\lim_{\theta\to 0} \dfrac{\sin 3\theta}{\sin 4\theta}$

30. $\lim_{t\to 0} \left(\dfrac{1}{\sin t} - \dfrac{1}{t}\right)$

31. $\lim_{x\to\infty} (1 + x)^{1/x}$

32. $\lim_{x\to\infty} \dfrac{x - 2x^2}{3x^2 + 5x}$

In Exercises 33–52, use l'Hospital's Rule to evaluate the limit.

33. $\lim_{\theta\to 0} \dfrac{\sin \theta^2}{\theta}$

34. $\lim_{t\to 1} \dfrac{t - 1}{\ln t - \sin \pi t}$

35. $\lim_{x\to\infty} \dfrac{\log_2 x}{\log_3 (x + 3)}$

36. $\lim_{y\to 0^+} \dfrac{\ln (y^2 + 2y)}{\ln y}$

37. $\lim_{y\to \pi/2} \left(\dfrac{\pi}{2} - y\right) \tan y$

38. $\lim_{x\to 0^+} (\ln x - \ln \sin x)$

39. $\lim_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{\sqrt{x}}\right)$

40. $\lim_{x\to 0} \left(\dfrac{1}{x^2}\right)^x$

41. $\lim_{x\to \pm\infty} \dfrac{3x - 5}{2x^2 - x + 2}$

42. $\lim_{x\to 0} \dfrac{\sin 7x}{\tan 11x}$

43. $\lim_{x\to\infty} (1 + 2x)^{1/(2 \ln x)}$

44. $\lim_{x\to (\pi/2)^-} (\cos x)^{\cos x}$

45. $\lim_{x\to 0^+} (1 + x)^{1/x}$

46. $\lim_{x\to 0^+} (\sin x)^{\tan x}$

47. $\lim_{x\to 1^+} x^{1/(1-x)}$

48. $\lim_{x\to\infty} \displaystyle\int_x^{2x} \frac{dt}{t}$

49. $\lim_{x\to 1} \dfrac{x^3 - 1}{4x^3 - x - 3}$

50. $\lim_{x\to\infty} \dfrac{2x^2 + 3x}{x^3 + x + 1}$

51. $\lim_{x\to 1} \dfrac{\int_1^x \cos t\, dt}{x^2 - 1}$

52. $\lim_{x\to 1} \dfrac{\int_1^x \frac{dt}{t}}{x^3 - 1}$

Group Activity In Exercises 53 and 54, do the following.

(a) **Writing to Learn** Explain why l'Hospital's Rule does not help you to find the limit.

(b) Use a graph to estimate the limit.

(c) Evaluate the limit analytically, using the techniques of Chapter 2.

53. $\lim_{x\to\infty} \dfrac{\sqrt{9x + 1}}{\sqrt{x + 1}}$

54. $\lim_{x\to \pi/2} \dfrac{\sec x}{\tan x}$

55. ***Continuous Extension*** Find a value of c that makes the function

$$f(x) = \begin{cases} \dfrac{9x - 3 \sin 3x}{5x^3}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

continuous at $x = 0$. Explain why your value of c works.

56. ***Continuous Extension*** Let $f(x) = |x|^x, x \neq 0$. Show that f has a removable discontinuity at $x = 0$ and extend the definition of f to $x = 0$ so that the extended function is continuous there.

57. ***Interest Compounded Continuously***

(a) Show that $\lim_{k\to\infty} A_0\left(1 + \dfrac{r}{k}\right)^{kt} = A_0 e^{rt}$.

(b) **Writing to Learn** Explain how the limit in part (a) connects interest compounded k times per year with interest compounded continuously.

58. ***L'Hospital's Rule*** Let

$$f(x) = \begin{cases} x + 2, & x \neq 0 \\ 0, & x = 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x + 1, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

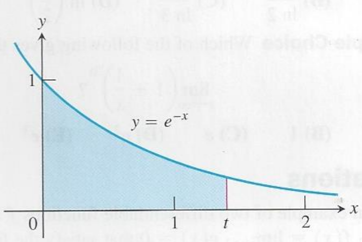
(a) Show that

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = 1 \quad \text{but} \quad \lim_{x\to 0} \frac{f(x)}{g(x)} = 2.$$

(b) **Writing to Learn** Explain why this does not contradict l'Hospital's Rule.

59. ***Solid of Revolution*** Let $A(t)$ be the area of the region in the first quadrant enclosed by the coordinate axes, the curve $y = e^{-x}$, and the line $x = t > 0$ as shown in the figure. Let $V(t)$ be the volume of the solid generated by revolving the region about the x-axis. Find the following limits.

(a) $\lim_{t\to\infty} A(t)$ **(b)** $\lim_{t\to\infty} \dfrac{V(t)}{A(t)}$ **(c)** $\lim_{t\to 0^+} \dfrac{V(t)}{A(t)}$



60. ***L'Hospital's Trap*** Let $f(x) = \dfrac{1 - \cos x}{x + x^2}$.

(a) Use graphs or tables to estimate $\lim_{x\to 0} f(x)$.

(b) Find the error in the following incorrect application of l'Hospital's Rule.

$$\lim_{x\to 0} \frac{1 - \cos x}{x + x^2} = \lim_{x\to 0} \frac{\sin x}{1 + 2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$$

61. ***Exponential Functions*** **(a)** Use the equation

$$a^x = e^{x \ln a}$$

to find the domain of

$$f(x) = \left(1 + \frac{1}{x}\right)^x.$$

(a) Find $\lim_{x\to -1^-} f(x)$.

(b) Find $\lim_{x\to -\infty} f(x)$.

Standardized Test Questions

62. True or False If $f(a) = g(a) = 0$ and $f'(a)$ and $g'(a)$ exist, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$ Justify your answer.

63. True or False $\lim_{x\to 0^+} x^x$ does not exist. Justify your answer.

64. Multiple Choice Which of the following gives the value of

$$\lim_{x\to 0} \frac{x}{\tan x}?$$

(A) -1 (B) 0 (C) 1 (D) π (E) Does not exist

65. Multiple Choice Which of the following gives the value of

$$\lim_{x\to 1} \frac{1 - \frac{1}{x}}{1 - \frac{1}{x^2}}?$$

(A) Does not exist (B) 2 (C) 1 (D) $1/2$ (E) 0

66. Multiple Choice Which of the following gives the value of

$$\lim_{x\to\infty} \frac{\log_2 x}{\log_3 x}?$$

(A) 1 (B) $\frac{\ln 3}{\ln 2}$ (C) $\frac{\ln 2}{\ln 3}$ (D) $\ln\left(\frac{3}{2}\right)$ (E) $\ln\left(\frac{2}{3}\right)$

67. Multiple Choice Which of the following gives the value of

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{3x}?$$

(A) 0 (B) 1 (C) e (D) e^2 (E) e^3

Explorations

68. Give an example of two differentiable functions f and g with $\lim_{x\to 3} f(x) = \lim_{x\to 3} g(x) = 0$ that satisfy the following.

(a) $\lim_{x\to 3} \frac{f(x)}{g(x)} = 7$ (b) $\lim_{x\to 3} \frac{f(x)}{g(x)} = 0$

(c) $\lim_{x\to 3} \frac{f(x)}{g(x)} = \infty$

69. Give an example of two differentiable functions f and g with $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$ that satisfy the following.

(a) $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 3$ (b) $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$

(c) $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$

Extending the Ideas

70. *Grapher Precision* Let $f(x) = \dfrac{1 - \cos x^6}{x^{12}}$.

(a) Explain why some graphs of f may give false information about $\lim_{x\to 0} f(x)$. Try the window $[-0.3, 0.3]$ by $[-0.5, 1]$. Illustrate your answer with a graph.

(b) Explain why tables may give false information about $\lim_{x\to 0} f(x)$. (*Hint:* Try tables with increments of 0.01.)

(c) Use l'Hospital's Rule to find $\lim_{x\to 0} f(x)$.

(d) **Writing to Learn** This is an example of a function for which graphers do not have enough precision to give reliable information. Explain this statement in your own words.

71. *Cauchy's Mean Value Theorem* Suppose that functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and suppose also that $g' \neq 0$ throughout (a, b). Then there exists a number c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Find all values of c in (a, b) that satisfy this property for the following given functions and intervals.

(a) $f(x) = x^3 + 1$, $g(x) = x^2 - x$, $[a, b] = [-1, 1]$

(b) $f(x) = \cos x$, $g(x) = \sin x$, $[a, b] = [0, \pi/2]$

72. *Why 0^∞ and $0^{-\infty}$ Are Not Indeterminate Forms* Assume that $f(x)$ is nonnegative in an open interval containing c and $\lim_{x\to 0} f(x) = 0$.

(a) If $\lim_{x\to c} g(x) = \infty$, show that $\lim_{x\to c} f(x)^{g(x)} = 0$.

(b) If $\lim_{x\to c} g(x) = -\infty$, show that $\lim_{x\to c} f(x)^{g(x)} = \infty$.