7.1. Exact Differential Equations

Just integrate to find the solution...

Example #1. Solve $\frac{dy}{dx} = 4x^3 + \sin x$ for $y = y(x)$ subject to the initial condition $y(0) = 4$.

SOLUTION: $dy = (4x^3 + \sin x)\,dx$. Integrate from the initial state to some later state $\Rightarrow$ $\int_4^y dz = \int_0^x (4t^3 + \sin t)\,dt$,

$[z]_4^y = [t^4 - \cos t]_0^x$, $y - 4 = x^4 - \cos x + 1$, $y = x^4 + 5 - \cos x$ ◄

Example #2. Solve $\frac{d^2y}{dx^2} = 42x^5 + e^x$ for $y = y(x)$ subject to the initial conditions $y'(0) = 7$ and $y(0) = 5$.

SOLUTION: $\frac{dy'}{dx} = 42x^5 + e^x$, $dy' = (42x^5 + e^x)\,dx$. Integrate from the initial state to some later state $\Rightarrow$ $\int_7^{y'} dz = \int_0^x (42t^5 + e^t)\,dt$,

$[z]_7^{y'} = [7t^6 + e^t]_0^x$, $y' - 7 = 7x^6 + e^x - 1$, $y' = \frac{dy}{dx} = 7x^6 + 6 + e^x$

$dy = (7x^6 + 6 + e^x)\,dx$. Integrate from the initial state to some later state $\Rightarrow$ $\int_5^y dz = \int_0^x (7t^6 + 6 + e^t)\,dt$, $[z]_5^y = [t^7 + 6t + e^t]_0^x$,

$y - 5 = x^7 + 6x + e^x - 1$, $y = x^7 + 6x + 4 + e^x$ ◄

CLASS WORK Solve the differential equations subjected to the indicated initial conditions.

(1) $\dfrac{dy}{dx} = 5x^4 + \cos x$, $y(0) = 8$ (2) $\dfrac{d^2y}{dx^2} = 12x^2 + e^{-x}$, $y'(0) = -3$, $y(0) = 5$

SOLUTIONS

(1) $dy = (5x^4 + \cos x)\,dx$, $\displaystyle\int_8^y dz = \int_0^x (5t^4 + \cos t)\,dt$, $[z]_8^y = [t^5 + \sin t]_0^x$,

$y - 8 = x^5 + \sin x$, $y = x^5 + 8 + \sin x$

(2) $\dfrac{dy'}{dx} = 12x^2 + e^{-x}$, $dy' = (12x^2 + e^{-x})\,dx$, $\displaystyle\int_{-3}^{y'} dz = \int_0^x (12t^2 + e^{-t})\,dt$,

$[z]_{-3}^{y'} = [4t^3 - e^{-t}]_0^x$, $y' + 3 = 4x^3 - e^{-x} + 1$, $y' = \dfrac{dy}{dx} = 4x^3 - 2 - e^{-x}$

$dy = (4x^3 - 2 - e^{-x})\,dx$, $\displaystyle\int_5^y dz = \int_0^x (4t^3 - 2 - e^{-t})\,dt$,

$[z]_5^y = [t^4 - 2t + e^{-t}]_0^x$, $y - 5 = x^4 - 2x + e^{-x} - 1$, $y = x^4 - 2x + 4 + e^{-x}$