7.1. The Simple First-Order Linear Differential Equation

$$\boxed{\frac{dy}{dx} + ky = f(x)} \qquad k \equiv \text{constant}$$

$$e^{kx}\frac{dy}{dx} + e^{kx}\cdot ky = e^{kx} f(x), \quad \text{Let } u = e^{kx}y,$$

$$\frac{du}{dx} = ke^{kx}y + e^{kx}\frac{dy}{dx} \Rightarrow \frac{du}{dx} = e^{kx}f(x), \quad du = e^{kx}f(x)\,dx,$$

so it can be integrated.

Example. Solve $\frac{dy}{dx} - y = \sin x$ for $y = y(x)$ subject to the initial condition $y(0) = 5$.

SOLUTION:

$$k = -1 \Rightarrow e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}\sin x, \quad \text{Let } u = e^{-x}y \Rightarrow$$

$$\frac{du}{dx} = -e^{-x}y + e^{-x}\frac{dy}{dx}, \quad \frac{du}{dx} = e^{-x}\sin x, \quad du = e^{-x}\sin x\,dx$$

From the worksheet $\int e^{-x}\sin x\,dx = -\frac{1}{2}e^{-x}(\sin x + \cos x)$

$y(0) = 5 \Rightarrow u_0 = e^{-0}\cdot 5 = 5$. So, integrate from the initial state to some later state...

$$\int_{u_0}^{u} dv = [v]_{u_0}^{u} = [v]_5^{e^{-x}y} = \int_0^x e^{-t}\sin t\,dt = -\frac{1}{2}\left[e^{-t}(\sin t + \cos t)\right]_0^x,$$

$$e^{-x}y - 5 = -\frac{1}{2}\left[e^{-x}(\sin x + \cos x) - 1\right] = \frac{1}{2} - \frac{1}{2}e^{-x}(\sin x + \cos x)$$

$$e^{-x}y = \frac{11}{2} - \frac{1}{2}e^{-x}(\sin x + \cos x)$$

$$y = \frac{11}{2}e^{x} - \frac{1}{2}(\sin x + \cos x)$$