7.2. Substitution in Definite Integrals

Examples. Use the indicated substitutions to evaluate the integral.

(1) $\int_0^4 \sin\left(\frac{\pi x}{8}\right) dx$, $u = \frac{\pi x}{8}$ SOLUTION: $x=0 \Rightarrow u=0$, $x=4 \Rightarrow u=\frac{\pi}{2}$

$\frac{du}{dx} = \frac{\pi}{8}$; $du = \frac{\pi}{8}dx$, $dx = \frac{8}{\pi}du$, $\int_{x=0}^{x=4} \sin\left(\frac{\pi x}{8}\right) dx = \int_{u=0}^{u=\pi/2} \sin u \cdot \frac{8}{\pi} du = \frac{8}{\pi}\int_0^{\pi/2} \sin u\, du =$

$= \frac{8}{\pi}\Big[-\cos u\Big]_0^{\pi/2} = \frac{8}{\pi}\cdot 1 = \frac{8}{\pi} = 2.546\ 479\ 089 \blacktriangleleft$

(2) $\int_0^{\pi/4} x\sec^2(x^2)\, dx$, $u = x^2$ SOLUTION: $x=0 \Rightarrow u=0$, $x=\frac{\pi}{4} \Rightarrow u=\frac{\pi^2}{16}$

$\frac{du}{dx} = 2x$, $du = 2x\,dx$, $dx = \frac{du}{2x}$, $\int_{x=0}^{x=\pi/4} x\sec^2(x^2)\, dx = \int_{u=0}^{u=\pi^2/16} \cancel{x}\sec^2 u \cdot \frac{du}{2\cancel{x}} =$

$= \frac{1}{2}\int_0^{\pi^2/16} \sec^2 u\, du = \frac{1}{2}\Big[\tan u\Big]_0^{\pi^2/16} = \frac{1}{2}\tan\left(\frac{\pi^2}{16}\right) = 0.354\ 582\ 3132 \blacktriangleleft$

(3) $\int_0^2 x\sqrt{4-x^2}\, dx$, $u = 4-x^2$ SOLUTION: $x=0 \Rightarrow u=4$, $x=2 \Rightarrow u=0$

$\frac{du}{dx} = -2x$, $du = -2x\,dx$, $dx = -\frac{du}{2x}$, $\int_{x=0}^{x=2} x\sqrt{4-x^2}\, dx = \int_{u=4}^{u=0} \cancel{x}\sqrt{u}\cdot -\frac{du}{2\cancel{x}} =$

$= \frac{1}{2}\int_0^4 \sqrt{u}\, du = \left[\frac{1}{3}\sqrt{u^3}\right]_0^4 = 2\frac{2}{3} \blacktriangleleft$

(4) $\int_0^{\pi/3} \tan x\, dx$; $u = \cos x$ SOLUTION: $x=0 \Rightarrow u=1$, $x=\frac{\pi}{3} \Rightarrow u=\frac{1}{2}$

$\frac{du}{dx} = -\sin x$, $du = -\sin x\, dx$, $dx = -\frac{du}{\sin x}$, $\int_{x=0}^{x=\pi/3} \tan x\, dx = \int_{x=0}^{x=\pi/3} \frac{\sin x}{\cos x} dx =$

$= \int_{u=1}^{u=1/2} \frac{\cancel{\sin x}}{u}\cdot -\frac{du}{\cancel{\sin x}} = \int_{1/2}^{1} \frac{du}{u} = \Big[\ln|u|\Big]_{1/2}^{1} = \ln 2 = 0.693\ 147\ 1806 \blacktriangleleft$

7.2. Substitution in Definite Integrals

CLASS WORK. Use the indicated substitutions to evaluate the integrals.

(1) $\int_0^1 \cos\left(\frac{\pi x}{2}\right) dx$, $u = \frac{\pi x}{2}$

(2) $\int_0^{\frac{\pi}{2}} x \sin(x^2)\, dx$, $u = x^2$

(3) $\int_0^2 \frac{x}{\sqrt{9-x^2}}\, dx$, $u = 9 - x^2$

(4) $\int_{\frac{\pi}{4}}^{\pi/2} \cot x\, dx$, $u = \sin x$

SOLUTIONS

(1) $x=0 \Rightarrow u=0$, $x=1 \Rightarrow u=\pi/2$, $\frac{du}{dx} = \frac{\pi}{2}$, $du = \frac{\pi}{2} dx$, $dx = \frac{2}{\pi} du$,

$$\int_{x=0}^{x=1} \cos\left(\frac{\pi x}{2}\right) dx = \int_{u=0}^{u=\pi/2} \cos u \cdot \frac{2}{\pi} du = \frac{2}{\pi}\int_0^{\pi/2} \cos u\, du = \frac{2}{\pi}\Big[\sin u\Big]_0^{\pi/2} = \frac{2}{\pi}\cdot 1 = \frac{2}{\pi} =$$
$$= 0.636\ 619\ 7724 \blacktriangleleft$$

(2) $x=0 \Rightarrow u=0$, $x=\frac{\pi}{2} \Rightarrow u=\pi^2/4$, $\frac{du}{dx} = 2x$, $du = 2x\, dx$, $dx = \frac{du}{2x}$,

$$\int_{x=0}^{x=\pi/2} x\sin(x^2)\, dx = \int_{u=0}^{u=\pi^2/4} \cancel{x} \sin u \cdot \frac{du}{2\cancel{x}} = \frac{1}{2}\int_0^{\pi^2/4} \sin u\, du = \frac{1}{2}\Big[-\cos u\Big]_0^{\pi^2/4} = \frac{1}{2}\left[1 - \cos\frac{\pi^2}{4}\right] =$$
$$= 0.890\ 605\ 9461 \blacktriangleleft$$

(3) $x=0 \Rightarrow u=9$, $x=2 \Rightarrow u=5$, $\frac{du}{dx} = -2x$, $du = -2x\, dx$, $dx = -\frac{du}{2x}$,

$$\int_{x=0}^{x=2} \frac{x}{\sqrt{9-x^2}}\, dx = \int_{u=9}^{u=5} \frac{\cancel{x}}{\sqrt{u}} \cdot -\frac{du}{2\cancel{x}} = \frac{1}{2}\int_5^9 \frac{du}{\sqrt{u}} = \frac{1}{2}\Big[2\sqrt{u}\Big]_5^9 = 3 - \sqrt{5} =$$
$$= 0.763\ 932\ 0225 \blacktriangleleft$$

(4) $x = \frac{\pi}{4} \Rightarrow u = \frac{1}{\sqrt{2}}$, $x = \frac{\pi}{2} \Rightarrow u = 1$, $\frac{du}{dx} = \cos x$, $du = \cos x\, dx$, $dx = \frac{du}{\cos x}$

$$\int_{\pi/4}^{\pi/2} \cot x\, dx = \int_{x=\pi/4}^{x=\pi/2} \frac{\cos x}{\sin x}\, dx = \int_{u=1/\sqrt{2}}^{u=1} \frac{\cancel{\cos x}}{u} \cdot \frac{du}{\cancel{\cos x}} = \int_{1/\sqrt{2}}^{1} \frac{du}{u} = \Big[\ln|u|\Big]_{1/\sqrt{2}}^{1} =$$

$$= \ln\sqrt{2} = 0.346\ 573\ 5903 \blacktriangleleft$$