AP CALCULUS AB

TRIGONOMETRIC SUBSTITUTIONS

For problems **1** and **2**, verify the indefinite integrals by differentiation.

1)

$$\int \sec\theta \tan^2\theta \, d\theta = \int f'(\theta)\,d\theta =$$

$$= \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta| = f(\theta)$$

Hint: use $\tan^2\theta + 1 = \sec^2\theta$.

$$f'(\theta) = \frac{1}{2}\left[\sec\theta\tan\theta\tan\theta + \sec\theta\sec^2\theta\right]$$

$$-\frac{1}{2}\cdot\frac{1}{(\sec\theta+\tan\theta)}\cdot(\sec\theta\tan\theta + \sec^2\theta) =$$

$$= \frac{1}{2}\sec\theta\left[\tan^2\theta + \sec^2\theta\right]$$

$$-\frac{1}{2}\,\frac{1}{\cancel{(\sec\theta+\tan\theta)}}\cdot\sec\theta\cancel{(\sec\theta+\tan\theta)}$$

$$= \frac{1}{2}\sec\theta\left[\tan^2\theta + \tan^2\theta \cancel{+1} \cancel{-1}\right] =$$

$$= \frac{1}{2}\sec\theta\left[2\tan^2\theta\right] = \sec\theta\tan^2\theta \;◄$$

2)

$$\int \cos^2\theta \, d\theta = \int f'(\theta)\,d\theta =$$

$$= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta = f(\theta)$$

Hint: use $\sin^2\theta + \cos^2\theta = 1$.

$$f'(\theta) = \frac{1}{2} + \frac{1}{2}\left[\cos\theta\cdot\cos\theta + \sin\theta\cdot -\sin\theta\right] =$$

$$= \frac{1}{2} + \frac{1}{2}\left[\cos^2\theta - \sin^2\theta\right] =$$

$$= \frac{1}{2} + \frac{1}{2}\left[\cos^2\theta - (1-\cos^2\theta)\right] =$$

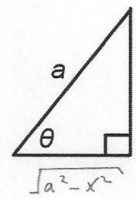
$$= \frac{1}{2}\left[\cancel{1} + \cos^2\theta \cancel{-1} + \cos^2\theta\right] =$$

$$= \frac{1}{\cancel{2}}\left[\cancel{2}\cos^2\theta\right] = \cos^2\theta \;◄$$

3) Use the triangle shown to make a trigonometric substitution to evaluate

$$\int \sqrt{a^2 - x^2}\,dx.$$

You will need the integral from problem **2**.



$$\cos\theta = \frac{\sqrt{a^2-x^2}}{a}$$

$$\sqrt{a^2-x^2} = a\cos\theta$$

$$\sin\theta = \frac{x}{a}, \quad x = a\sin\theta, \quad \frac{dx}{d\theta} = a\cos\theta,$$

$$dx = a\cos\theta\,d\theta$$

$$\int\sqrt{a^2-x^2}\,dx = \int a\cos\theta\cdot a\cos\theta\,d\theta =$$

$$= a^2\int\cos^2\theta\,d\theta = \frac{a^2}{2}\theta + \frac{a^2}{2}\sin\theta\cos\theta,$$

$$\theta = \sin^{-1}\left(\frac{x}{a}\right),$$

$$\int\sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

$$+\frac{a^{\cancel{2}}}{2}\cdot\frac{x}{\cancel{a}}\cdot\frac{\sqrt{a^2-x^2}}{\cancel{a}} =$$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} \;◄$$