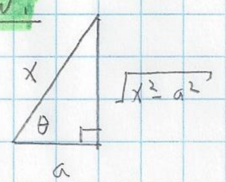
7.2. Trigonometric Substitutions

Useful for integrals containing $\sqrt{a^2-x^2}$ and $\sqrt{x^2-a^2}$.

Example. Evaluate $\int \sqrt{x^2-a^2}\,dx$.

SOLUTION:



$\tan\theta = \dfrac{\sqrt{x^2-a^2}}{a}$, $\cos\theta = \dfrac{a}{x}$, $\sec\theta = \dfrac{x}{a}$

$\sqrt{x^2-a^2} = a\tan\theta$, $x = a\sec\theta$, $\dfrac{dx}{d\theta} = a\sec\theta\tan\theta$

$dx = a\sec\theta\tan\theta\,d\theta$,

$$\int\sqrt{x^2-a^2}\,dx = \int a\tan\theta \cdot a\sec\theta\tan\theta\,d\theta = a^2\int\sec\theta\tan^2\theta\,d\theta.$$

From the worksheet: $\int\sec\theta\tan^2\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta| \Rightarrow$

$$\int\sqrt{x^2-a^2}\,dx = \frac{a^2}{2}\sec\theta\tan\theta - \frac{a^2}{2}\ln|\sec\theta+\tan\theta| =$$

$$= \frac{a^2}{2}\cdot\frac{x}{a}\cdot\frac{\sqrt{x^2-a^2}}{a} - \frac{a^2}{2}\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right| =$$

$$= \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\left[\ln\left|x+\sqrt{x^2-a^2}\right| - \ln|a|\right] =$$

can forget about this because it is a constant

$$= \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|$$