7.4. Exponential Growth and Decay

Exponential Growth, for example, population growth ...

$p \equiv$ population
$t \equiv$ time
$p_0 \equiv$ population at time $t=0$

The rate of increase of the population is proportional to the current population $\Rightarrow$

$$\boxed{\frac{dp}{dt} = kp}$$

$k \equiv$ proportionality constant, $k > 0$.

$\frac{dp}{p} = k\,dt$. Integrate from the initial state to some later state $\Rightarrow$

$$\int_{p_0}^{p} \frac{dz}{z} = k\int_0^t dx\,,\quad [\ln z]_{p_0}^{p} = k\,[x]_0^t\,,\quad \ln\left(\frac{p}{p_0}\right) = kt\,,\quad \frac{p}{p_0} = e^{kt}$$

$$\boxed{p = p_0 e^{kt}}$$

Example #1. In 1960, the population of France was 46.62 million people. In 2020, it was 67.39 million. Estimate the population of France in 2035.

SOLUTION:

$p = p_0 e^{kt}$, $p = 46.62\,e^{kt}$ $(1960 \rightarrow t=0)$,

$2020 \rightarrow t = 60$ $\quad 67.39 = 46.62\,e^{60k}$, $e^{60k} = \frac{67.39}{46.62}$, $60k = \ln\left(\frac{67.39}{46.62}\right)$,

$$k = \frac{1}{60}\ln\left(\frac{67.39}{46.62}\right) = 0.00614\ldots/\text{yr}$$

$2035 \rightarrow t = 75$, $p = 46.62\,e^{75k} = 73.89$ million

Exponential Decay, for example, radioactive decay...

$C \equiv$ concentration of a radioactive isotope
$t \equiv$ time
$C_0 \equiv$ concentration at time $t = 0$

The rate of decrease of the concentration is proportional to the current concentration $\Rightarrow$

$$\boxed{\frac{dc}{dt} = -kc}$$

$k \equiv$ proportionality constant, $k > 0$.

$\frac{dc}{c} = -k\,dt$. Integrate from the initial state to some later state $\Rightarrow$

$$\int_{c_0}^{c} \frac{dz}{z} = -k \int_0^t dx \,, \quad \left[\ln z\right]_{c_0}^{c} = -k \left[x\right]_0^t \,, \quad \ln\left(\frac{c}{c_0}\right) = -kt \,, \quad \frac{c}{c_0} = e^{-kt}$$

$$\boxed{c = c_0 e^{-kt}}$$

Example #2. A sample containing 100% of radioactive Barium-137 (Ba-137) will contain 80% of it original Ba-137 after 50 seconds. After how many seconds will the sample contain 40% of the Ba-137?

SOLUTION:

$$c = c_0 e^{-kt} \,, \quad 0.8\cancel{c_0} = \cancel{c_0} e^{-50k} \,, \quad -50k = \ln 0.8 \,, \quad k = -\frac{\ln 0.8}{50} = 0.00446/\text{sec}$$

$$0.4\cancel{c_0} = \cancel{c_0} e^{-kt} \,, \quad -kt = \ln 0.4 \,, \quad t = -\frac{\ln 0.4}{k} = 205.31 \text{ sec} = 3:25$$